Corner complex scaling for the interior transmission eigenvalue problem

Séminaire IDEFIX 27th April 2023

Anne-Sophie Bonnet-Ben Dhia[1], Lucas Chesnel[2], Florian Monteghetti[1]

[1]POEMS (CNRS-INRIA-ENSTA Paris), Palaiseau, France
[2]IDEFIX (EDF-INRIA-ENSTA Paris), Palaiseau, France
Contact: florian.monteghetti@ensta-paris.fr

Contents

1 Introduction
- Objective
- Motivation
- Basics of ITEP
- Outline

2 Strongly-oscillating singularities in ITEP

3 Corner complex scaling

4 Numerical results

5 Conclusion

Objective

Let $D \subset \mathbb{R}^d$ be an open bounded set, modeling a scatterer.

Interior Transmission Eigenvalue Problem (ITEP)

Find $(k, (u, w)) \in \mathbb{C} \times \left[H^1(D)\right]^2$, $(u, w) \neq 0$, such that

$$\nabla \cdot [A \nabla u] + k^2 n\, u = 0, \quad \Delta w + k^2 w = 0 \quad (x \in D),$$

with boundary conditions

$$u = w, \quad \partial_{\nu_A} u = \partial_\nu w \quad (x \in \partial D),$$

where $n \in L^\infty(D)$ and $A \in L^\infty(D, \mathbb{R}^{d \times d})$.

Definition: k is a transmission eigenvalue.

Interpretation: u is the total field and w is the incident field.

Objective

Let $D \subset \mathbb{R}^d$ be an open bounded set, modeling a scatterer.

Interior Transmission Eigenvalue Problem (ITEP)

Find $(k, (u, w)) \in \mathbb{C} \times \left[H^1(D)\right]^2$, $(u, w) \neq 0$, such that

$$\nabla \cdot [A \nabla u] + k^2 n\, u = 0, \quad \Delta w + k^2 w = 0 \quad (x \in D),$$

with boundary conditions

$$u = w, \quad \partial_{\nu_A} u = \partial_\nu w \quad (x \in \partial D),$$

where $n \in L^\infty(D)$ and $A \in L^\infty(D, \mathbb{R}^{d\times d})$.

Definition: k is a transmission eigenvalue.

Interpretation: u is the total field and w is the incident field.

Broad objective: Computation of k when A is such that

$$[H^1(D)]^2 \text{ is not the right functional space,}$$

since the problem is not Fredholm.

Next: basics of ITEP in $[H^1(D)]^2$

Motivation: why study transmission eigenvalues?

Transmission eigenvalues (TEs) are useful in inverse scattering.

Theorem. Faber-Krahn type estimate (Cakoni and Haddar 2012, Thm 3.5)

If $A = I$ and $\inf_{x \in D} n(x) > 1$, then $\|n\|_{L^\infty(D)} > \frac{\lambda_1(D)}{k_1^2(D)}$.

Motivation: why study transmission eigenvalues?

Transmission eigenvalues (TEs) are useful in inverse scattering.

Theorem. Faber-Krahn type estimate (Cakoni and Haddar 2012, Thm 3.5)

If $A = I$ and $\inf_{x \in D} n(x) > 1$, then $\|n\|_{L^\infty(D)} > \frac{\lambda_1(D)}{k_1^2(D)}$.

Definition. Space $\mathcal{H}$ of Herglotz wave functions:

$$u^i(\boldsymbol{x}) = \int_{\mathbb{S}^2} e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{y}}} g_i(\hat{\boldsymbol{y}})\,\mathrm{d}\sigma(\hat{\boldsymbol{y}}) \quad \text{with} \quad g_i \in L^2(\mathbb{S}^2).$$

Definition. Far-field operator:

$$F(k, D) : \mathcal{H} \ni u^i \mapsto u^s_\infty \in L^2(\mathbb{S}^2),$$

where u^s_∞ is the far-field pattern of the scattered field u^s.

Motivation: why study transmission eigenvalues?

Transmission eigenvalues (TEs) are useful in inverse scattering.

Theorem. Faber-Krahn type estimate (Cakoni and Haddar 2012, Thm 3.5)

If $A = I$ and $\inf_{x \in D} n(x) > 1$, then $\|n\|_{L^\infty(D)} > \frac{\lambda_1(D)}{k_1^2(D)}$.

Definition. Space $\mathcal{H}$ of Herglotz wave functions:

$$u^i(\boldsymbol{x}) = \int_{\mathbb{S}^2} e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{y}}} g_i(\hat{\boldsymbol{y}})\, \mathrm{d}\sigma(\hat{\boldsymbol{y}}) \quad \text{with} \quad g_i \in L^2(\mathbb{S}^2).$$

Definition. Far-field operator:

$$F(k, D) : \mathcal{H} \ni u^i \mapsto u^s_\infty \in L^2(\mathbb{S}^2),$$

where u^s_∞ is the far-field pattern of the scattered field u^s.

Theorem. Arbitrarily-small far field (Cakoni, Colton, and Haddar 2021)

If $(k, (u, w))$ solves the ITEP, then $\forall \varepsilon > 0$,

$$\exists u^i_\varepsilon \in \mathcal{H} : \|u - u^i_\varepsilon\|_{L^2(D)} \le \varepsilon \quad \text{and} \quad \|F(k, D)u^i_\varepsilon\|_{L^2(\mathbb{S}^2)} \lesssim \varepsilon.$$

- If k is a nonscattering wavenumber, we can achieve $\varepsilon = 0$.

Standard setting

Weak formulation in $V = \left\{(u,w) \in \left[H^1(D)\right]^2 \,|\, u - w \in H^1_0(D)\right\}$

Find $(k,(u,w)) \in \mathbb{C} \times V \backslash \{0\}$ such that $\forall\, (\varphi_u, \varphi_w) \in V$,

$$(A\nabla u, \nabla\varphi_u)_D - (\nabla w, \nabla\varphi_w)_D = k^2 \left[(n\, u, \varphi_u)_D - (w, \varphi_w)_D\right].$$

⚠ This is a sign-changing problem with complex eigenvalues!

Standard setting

Weak formulation in $V = \left\{(u,w) \in \left[H^1(D)\right]^2 \,\middle|\, u - w \in H_0^1(D)\right\}$

Find $(k,(u,w)) \in \mathbb{C} \times V \backslash \{0\}$ such that $\forall\,(\varphi_u,\varphi_w) \in V$,

$$(A\nabla u, \nabla\varphi_u)_D - (\nabla w, \nabla\varphi_w)_D = k^2\left[(n\,u,\varphi_u)_D - (w,\varphi_w)_D\right].$$

⚠ This is a sign-changing problem with complex eigenvalues!

Theorem (Bonnet-Ben Dhia, Chesnel, and Haddar 2011, Thm. 5.1)

Let $A \in L^\infty\left(D,\mathbb{R}^{3\times 3}\right)$ symmetric p.d. and $n \in L^\infty\left(D,\mathbb{R}_+\right)$.
If there is a neighborhood $\mathcal{N}$ of ∂D s.t.:

$$\inf_{x\in\mathcal{N}} (I - A(x)) > 0 \quad \text{and} \quad \inf_{x\in\mathcal{N}} (1 - n(x)) > 0,$$

then $\sigma_{\text{ITEP}} = \{k_n\}_{n\geq 1}$ with ∞ as the only possible accumulation point.

Proof:

Standard setting

Weak formulation in $V = \left\{(u,w) \in \left[H^1(D)\right]^2 \,\middle|\, u - w \in H^1_0(D)\right\}$

Find $(k,(u,w)) \in \mathbb{C} \times V\backslash\{0\}$ such that $\forall\, (\varphi_u, \varphi_w) \in V$,

$$(A\nabla u, \nabla\varphi_u)_D - (\nabla w, \nabla\varphi_w)_D = k^2 \left[(n\, u, \varphi_u)_D - (w, \varphi_w)_D\right].$$

⚠ This is a sign-changing problem with complex eigenvalues!

Theorem (Bonnet-Ben Dhia, Chesnel, and Haddar 2011, Thm. 5.1)

Let $A \in L^\infty\left(D, \mathbb{R}^{3\times3}\right)$ symmetric p.d. and $n \in L^\infty\left(D, \mathbb{R}_+\right)$.
If there is a neighborhood $\mathcal{N}$ of ∂D s.t.:

$$\inf_{x\in\mathcal{N}} (I - A(x)) > 0 \quad \text{and} \quad \inf_{x\in\mathcal{N}} (1 - n(x)) > 0,$$

then $\sigma_{\text{ITEP}} = \{k_n\}_{n\geq1}$ with ∞ as the only possible accumulation point.

Proof: Bilinear form $a_k : V \times V \to \mathbb{C}$:

$$a_k(\boldsymbol{U}, \boldsymbol{\Phi}) := (A\nabla u, \nabla\varphi_u)_D - (\nabla w, \nabla\varphi_w)_D + \Im(k)^2 \left[(n\, u, \varphi_u)_D - (w, \varphi_w)_D\right] + b_k(\boldsymbol{U}, \boldsymbol{\Phi}),$$

Standard setting

Weak formulation in $V = \left\{(u,w) \in \left[H^1(D)\right]^2 \mid u - w \in H_0^1(D)\right\}$

Find $(k,(u,w)) \in \mathbb{C} \times V\backslash\{0\}$ such that $\forall\, (\varphi_u, \varphi_w) \in V$,

$$(A\nabla u, \nabla\varphi_u)_D - (\nabla w, \nabla\varphi_w)_D = k^2 \left[(n\, u, \varphi_u)_D - (w, \varphi_w)_D\right].$$

⚠ This is a sign-changing problem with complex eigenvalues!

Theorem (Bonnet-Ben Dhia, Chesnel, and Haddar 2011, Thm. 5.1)

Let $A \in L^\infty(D, \mathbb{R}^{3\times 3})$ symmetric p.d. and $n \in L^\infty(D, \mathbb{R}_+)$.
If there is a neighborhood $\mathcal{N}$ of ∂D s.t.:

$$\inf_{x\in\mathcal{N}} (I - A(x)) > 0 \quad \text{and} \quad \inf_{x\in\mathcal{N}} (1 - n(x)) > 0,$$

then $\sigma_{\text{ITEP}} = \{k_n\}_{n\geq 1}$ with ∞ as the only possible accumulation point.

Proof: Bilinear form $a_k : V \times V \to \mathbb{C}$:

$$a_k(\boldsymbol{U}, \boldsymbol{\Phi}) := \underbrace{(A\nabla u, \nabla\varphi_u)_D - (\nabla w, \nabla\varphi_w)_D + \Im(k)^2 \left[(n\, u, \varphi_u)_D - (w, \varphi_w)_D\right]}_{T\text{-coercive}} + \underbrace{b_k(\boldsymbol{U}, \boldsymbol{\Phi})}_{\text{compact}},$$

with $T(u,w) = (u, -w) + R_{\mathcal{N}}(u,w)$

Standard setting

Weak formulation in $V = \left\{(u,w) \in \left[H^1(D)\right]^2 \mid u - w \in H_0^1(D)\right\}$

Find $(k,(u,w)) \in \mathbb{C} \times V \setminus \{0\}$ such that $\forall\, (\varphi_u, \varphi_w) \in V$,

$$(A\nabla u, \nabla\varphi_u)_D - (\nabla w, \nabla \varphi_w)_D = k^2 \left[(n\,u, \varphi_u)_D - (w, \varphi_w)_D\right].$$

⚠ This is a sign-changing problem with complex eigenvalues!

Theorem (Bonnet-Ben Dhia, Chesnel, and Haddar 2011, Thm. 5.1)

Let $A \in L^\infty\left(D, \mathbb{R}^{3\times 3}\right)$ symmetric p.d. and $n \in L^\infty\left(D, \mathbb{R}_+\right)$.
If there is a neighborhood $\mathcal{N}$ of ∂D s.t.:

$$\inf_{x\in\mathcal{N}} (I - A(x)) > 0 \quad \text{and} \quad \inf_{x\in\mathcal{N}} (1 - n(x)) > 0,$$

then $\sigma_{\text{ITEP}} = \{k_n\}_{n\geq 1}$ with ∞ as the only possible accumulation point.

Proof: Bilinear form $a_k : V \times V \to \mathbb{C}$:

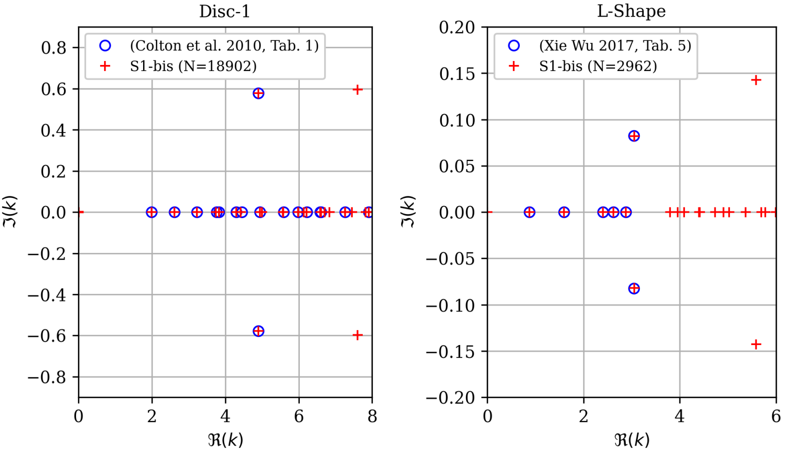
$$a_k(\boldsymbol{U}, \boldsymbol{\Phi}) := \underbrace{(A\nabla u, \nabla\varphi_u)_D - (\nabla w, \nabla\varphi_w)_D + \Im(k)^2 \left[(n\,u, \varphi_u)_D - (w, \varphi_w)_D\right]}_{T\text{-coercive}} + \underbrace{b_k(\boldsymbol{U}, \boldsymbol{\Phi})}_{\text{compact}},$$

with $T(u,w) = (u, -w) + R_{\mathcal{N}}(u,w)$ where $R_{\mathcal{N}}(u,w) = \chi_{\mathcal{N}}\,(-2w, 0)$.

Numerical results

Approximation space: $V_h \subset V = H^1(\Omega) \times H_0^1(\Omega)$, isoparametric Lagrange elements of degree p.

Implementation: gmsh / fenicsx / PETSc / SLEPc.



(a) Disc-1 ($p = 2$). (b) L-shape ($p = 1$).

Outline

Objective: Computation of transmission eigenvalues k when

$$I - A(x) \text{ changes sign around } \boldsymbol{x_t} \in \partial D,$$

leading to a non-Fredholm problem in $H^1(D)$.

Outline

2. Strongly-oscillating singularities in ITEP
   - Why do we lose Fredholmness?
   - Can we recover it in an extended space?
3. Corner complex scaling
   - How can we approximate this extended space?
4. Numerical results
   - Does this actually works?

Contents

1 Introduction

2 Strongly-oscillating singularities in ITEP
- Problem setting
- Local singularity analysis
- A new functional setting
- Numerical illustration

3 Corner complex scaling

4 Numerical results

5 Conclusion

Problem setting

Find $(k,(u,w)) : \nabla \cdot [A \nabla u] + k^2 n\, u = 0, \quad \Delta w + k^2 w = 0 \quad (x \in D),$

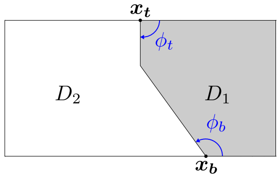
$$u = w, \qquad \partial_{\nu_A} u = \partial_\nu w \qquad (x \in \partial D).$$

Case of interest

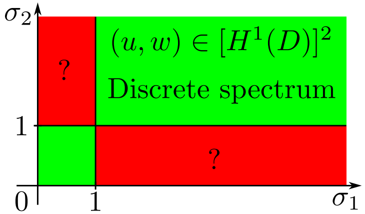
$$A(x) = \sigma_1 \mathbb{1}_{D_1}(x) + \sigma_2 \mathbb{1}_{D_2}(x),$$

$$n(x) > 1,$$

where $\sigma_i > 0$.



⚠ Nature of spectrum depends upon (σ_1, σ_2):



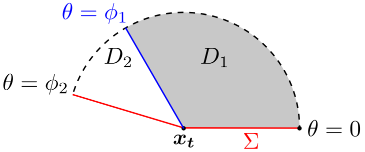
Next: singularity analysis around $x_{t/b}$.

Local singularity analysis (1/2)

Local interior transmission problem

$$\begin{cases} \nabla \cdot [A\nabla u] + \cancel{k^2 n u} = 0 & (D) \\ \Delta w + \cancel{k^2 w} = 0 & (D) \\ u - w = 0,\ \partial_{n_A} u - \partial_n w = 0 & (\Sigma), \end{cases}$$

where $A(\theta) = \sigma_1 \mathbb{1}_{D_1} + \sigma_2 \mathbb{1}_{D_2}$.

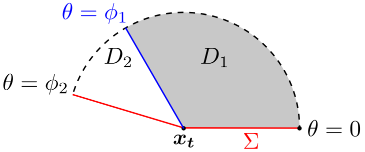


Local singularity analysis (1/2)

Local interior transmission problem

$$\begin{cases} \nabla \cdot [A \nabla u] + \cancel{k^2 n u} = 0 & (D) \\ \Delta w + \cancel{k^2 w} = 0 & (D) \\ u - w = 0,\ \partial_{n_A} u - \partial_n w = 0 & (\Sigma), \end{cases}$$

where $A(\theta) = \sigma_1 \mathbb{1}_{D_1} + \sigma_2 \mathbb{1}_{D_2}$.



Proposition (Dispersion relation).

Local solutions have the form

$$\begin{bmatrix} u \\ w \end{bmatrix} (r, \theta) = \begin{bmatrix} \Phi_0^u(\theta) \\ \Phi_0^w(\theta) \end{bmatrix} + \sum_{\eta \in H(\sigma_1, \sigma_2)} r^{i\eta} \begin{bmatrix} a_\eta \Phi_\eta^u(\theta) \\ b_\eta \Phi_\eta^w(\theta) \end{bmatrix},$$

where $(\Phi_\eta^u, \Phi_\eta^w) \in [H^1_{\text{per}}(-\pi, \pi)]^2$ and $H(\sigma_1, \sigma_2)$ is:

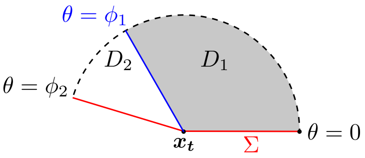
⚠ $\eta \in \mathbb{R}^* \Leftrightarrow$ strongly-oscillating singularities $r^{i\eta} \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) \notin [H^1(D)]^2$.

Local singularity analysis (1/2)

Local interior transmission problem

$$\begin{cases} \nabla \cdot [A \nabla u] + \cancel{k^2 n u} = 0 & (D) \\ \Delta w + \cancel{k^2 w} = 0 & (D) \\ u - w = 0,\ \partial_{n_A} u - \partial_n w = 0 & (\Sigma), \end{cases}$$

where $A(\theta) = \sigma_1 \mathbb{1}_{D_1} + \sigma_2 \mathbb{1}_{D_2}$.



Proposition (Dispersion relation).

Local solutions have the form

$$\begin{bmatrix} u \\ w \end{bmatrix} (r, \theta) = \begin{bmatrix} \Phi_0^u(\theta) \\ \Phi_0^w(\theta) \end{bmatrix} + \sum_{\eta \in H(\sigma_1, \sigma_2)} r^{i\eta} \begin{bmatrix} a_\eta \Phi_\eta^u(\theta) \\ b_\eta \Phi_\eta^w(\theta) \end{bmatrix},$$

where $(\Phi_\eta^u, \Phi_\eta^w) \in [H^1_{\text{per}}(-\pi, \pi)]^2$ and $H(\sigma_1, \sigma_2)$ is:

$$H(\sigma_1, \sigma_2) := \{\eta \in \mathbb{C}^* \mid \det \mathfrak{M}(\eta, \sigma_1, \sigma_2) = 0\},$$

with $\mathfrak{M} \in \mathbb{C}^{6\times 6}$.

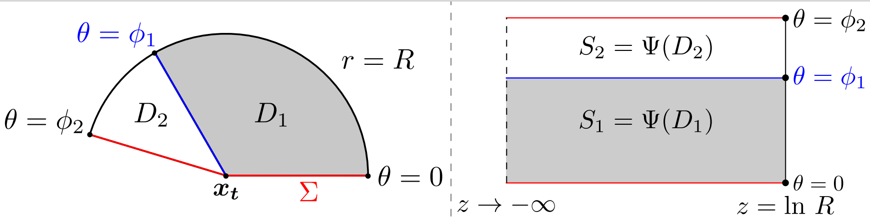
⚠ $\eta \in \mathbb{R}^* \Leftrightarrow$ strongly-oscillating singularities $r^{i\eta} \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) \notin [H^1(D)]^2$.

Local singularity analysis (2/2)

⚠ $\exists \eta \in \mathbb{R}^* : \det \mathfrak{M}(\eta, \sigma_1, \sigma_2) = 0 \Leftrightarrow$ strongly-oscillating singularity:

$$r^{i\eta}\, \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) \in [L^2(D) \backslash H^1(D)]^2.$$

Euler coordinates $(z, \theta) := (\ln r, \theta)$

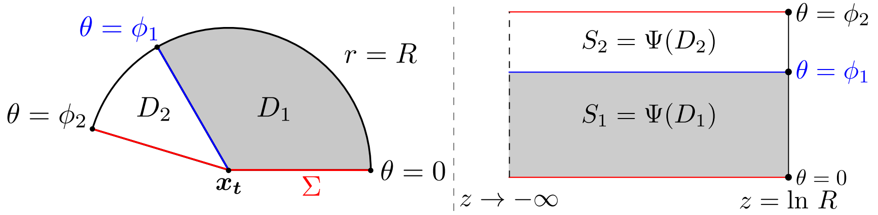


Local singularity analysis (2/2)

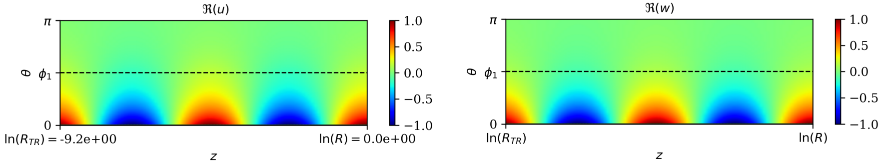
⚠ $\exists \eta \in \mathbb{R}^* : \det \mathfrak{M}(\eta, \sigma_1, \sigma_2) = 0 \Leftrightarrow$ strongly-oscillating singularity:

$$r^{i\eta}\, \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) \in [L^2(D) \backslash H^1(D)]^2.$$

Euler coordinates $(z, \theta) := (\ln r, \theta)$



Plot of singularity for $(\phi_1, \phi_2) = (\pi/2, \pi)$ and $(\sigma_1, \sigma_2) = (1.01, 0.5)$:



New functional settings for ITEP (Bonnet-Ben Dhia and Chesnel 2013)

The local analysis suggests defining the singularity region:

$$\mathscr{R} := \left\{ (\sigma_1, \sigma_2) \in \mathbb{R}^2 \,|\, \exists \eta \in \mathbb{R}^* : \det \mathfrak{M}_{\phi_1,\phi_2}(\eta, \sigma_1, \sigma_2) = 0 \right\}.$$

⚠ $(\sigma_1, \sigma_2) \in \mathscr{R} \Leftrightarrow$ strongly-oscillating $r^{i\eta}\, \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) \in [L^2(D) \backslash H^1(D)]^2$.

$\Leftrightarrow$ Fredholmness is lost in $H^1(D)$.

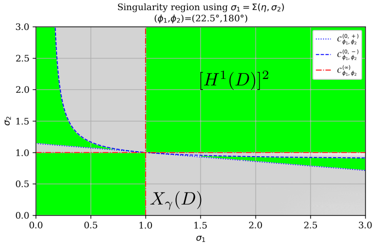
New functional settings for ITEP (Bonnet-Ben Dhia and Chesnel 2013)

The local analysis suggests defining the singularity region:

$$\mathscr{R} := \left\{ (\sigma_1, \sigma_2) \in \mathbb{R}^2 \,|\, \exists \eta \in \mathbb{R}^* : \det \mathfrak{M}_{\phi_1,\phi_2}(\eta, \sigma_1, \sigma_2) = 0 \right\}.$$

⚠ $(\sigma_1, \sigma_2) \in \mathscr{R} \Leftrightarrow$ strongly-oscillating $r^{i\eta}\, \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) \in [L^2(D)\backslash H^1(D)]^2$.
$\Leftrightarrow$ Fredholmness is lost in $H^1(D)$.

Functional spaces that achieves Fredholmness



New functional settings for ITEP (Bonnet-Ben Dhia and Chesnel 2013)

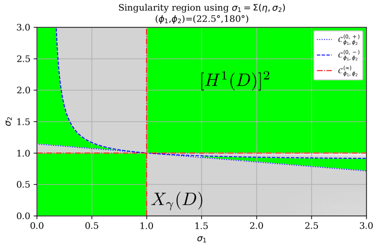
The local analysis suggests defining the singularity region:

$$\mathscr{R} := \left\{ (\sigma_1, \sigma_2) \in \mathbb{R}^2 \,|\, \exists \eta \in \mathbb{R}^* : \det \mathfrak{M}_{\phi_1,\phi_2}(\eta, \sigma_1, \sigma_2) = 0 \right\}.$$

⚠ $(\sigma_1, \sigma_2) \in \mathscr{R} \Leftrightarrow$ strongly-oscillating $r^{i\eta}\, \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) \in [L^2(D) \backslash H^1(D)]^2$.

$\Leftrightarrow$ Fredholmness is lost in $H^1(D)$.

Functional spaces that achieves Fredholmness



For any $\gamma \in \mathbb{R}$,

$$X_\gamma(D) := \text{span}\{r^{i\eta}\boldsymbol{\Phi}_{\boldsymbol{\eta}} + \gamma\, r^{-i\eta}\boldsymbol{\Phi}_{-\boldsymbol{\eta}}\} \oplus V,$$

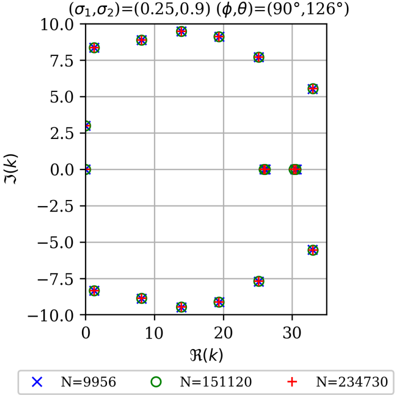
where $V \subset H^1(D)$ is a Kondratiev space.

Next: can we use H^1-FEM when $(\sigma_1, \sigma_2) \in \mathscr{R}$?

Numerical illustration of lack of convergence

Computation using H^1-FEM for increasing N (# of DoF):

Fredholm in $H^1(D)$

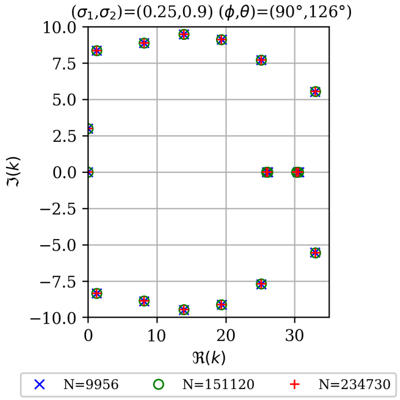


Fredholm in $X_\gamma(D)$

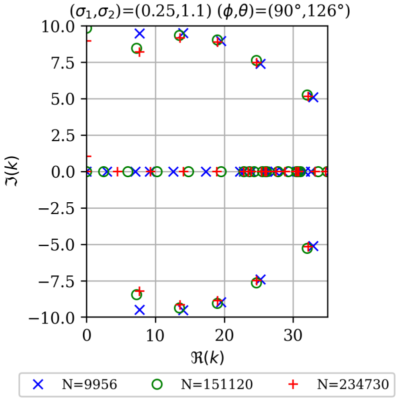
Numerical illustration of lack of convergence

Computation using H^1-FEM for increasing N (# of DoF):

Fredholm in $H^1(D)$



Fredholm in $X_\gamma(D)$



Next: how can we discretize in $X_\gamma(D)$?

Contents

1. Introduction
2. Strongly-oscillating singularities in ITEP
3. Corner complex scaling
    - Definition
    - Validation
4. Numerical results
5. Conclusion

Corner complex scaling: principle

Principle. Let $\alpha \in \mathbb{C}$. Define a new "ITEPα" such that:

k is an $X_\gamma(D)$-eigenvalue of ITEP $\Longleftrightarrow$ k is a $H^1(D)$-eigenvalue of ITEPα .

Assume $\gamma = 0$ and let $(u, w) \in X_\gamma(D)$. Intuitively, we would like

$$\text{(ITEP)} \qquad (u, w) \underset{r=|\boldsymbol{x}-\boldsymbol{x_t}|\to 0}{\sim} e^{i\eta \ln r}\, \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) + \boldsymbol{c_0} \qquad (\Im(\eta) = 0)$$

$$\downarrow$$

$$\text{(ITEP}\alpha\text{)} \qquad (u_\alpha, w_\alpha) \underset{r=|\boldsymbol{x}-\boldsymbol{x_t}|\to 0}{\sim} e^{i\frac{\eta}{\alpha} \ln r}\, \boldsymbol{\Phi}_{\boldsymbol{\eta}}(\theta) + \boldsymbol{c_0} \qquad \left(\Im\left(\frac{\eta}{\alpha}\right) < 0\right)$$

Corner complex scaling: principle

Principle. Let $\alpha \in \mathbb{C}$. Define a new "ITEPα" such that:

k is an $X_\gamma(D)$-eigenvalue of ITEP $\iff$ k is a $H^1(D)$-eigenvalue of ITEPα .

Assume $\gamma = 0$ and let $(u, w) \in X_\gamma(D)$. Intuitively, we would like

$$\text{(ITEP)} \qquad (u, w) \underset{r=|\boldsymbol{x}-\boldsymbol{x_t}|\to 0}{\sim} e^{i\eta \ln r}\, \boldsymbol{\Phi_\eta}(\theta) + \boldsymbol{c_0} \qquad (\Im(\eta) = 0)$$

$$\downarrow$$

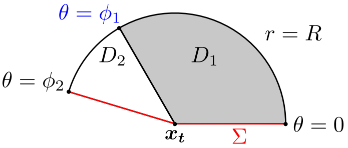
$$\text{(ITEP}\alpha\text{)} \qquad (u_\alpha, w_\alpha) \underset{r=|\boldsymbol{x}-\boldsymbol{x_t}|\to 0}{\sim} e^{i\frac{\eta}{\alpha} \ln r}\, \boldsymbol{\Phi_\eta}(\theta) + \boldsymbol{c_0} \qquad \left(\Im\left(\frac{\eta}{\alpha}\right) < 0\right)$$

Definition of ITEPα. Substitution

$$r\partial_r \to \alpha r \partial_r$$

around the corner.

(Bonnet-Ben Dhia, Carvalho, Chesnel, and Ciarlet 2016)



Corner complex scaling: principle

Principle. Let $\alpha \in \mathbb{C}$. Define a new "ITEPα" such that:

k is an $X_\gamma(D)$-eigenvalue of ITEP $\Longleftrightarrow$ k is a $H^1(D)$-eigenvalue of ITEPα .

Assume $\gamma = 0$ and let $(u, w) \in X_\gamma(D)$. Intuitively, we would like

$$\text{(ITEP)} \qquad (u, w) \underset{r=|\boldsymbol{x}-\boldsymbol{x_t}|\to 0}{\sim} e^{i\eta \ln r}\, \boldsymbol{\Phi_\eta}(\theta) + \boldsymbol{c_0} \qquad (\Im(\eta) = 0)$$

$$\downarrow$$

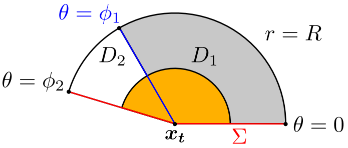
$$\text{(ITEP}\alpha\text{)} \quad (u_\alpha, w_\alpha) \underset{r=|\boldsymbol{x}-\boldsymbol{x_t}|\to 0}{\sim} e^{i\frac{\eta}{\alpha}\ln r}\, \boldsymbol{\Phi_\eta}(\theta) + \boldsymbol{c_0} \qquad \left(\Im\left(\frac{\eta}{\alpha}\right) < 0\right)$$

Definition of ITEPα. Substitution

$$r\partial_r \to \alpha r \partial_r$$

around the corner.

(Bonnet-Ben Dhia, Carvalho, Chesnel, and Ciarlet 2016)



Next: weak formulation?

Corner complex scaling: weak formulation

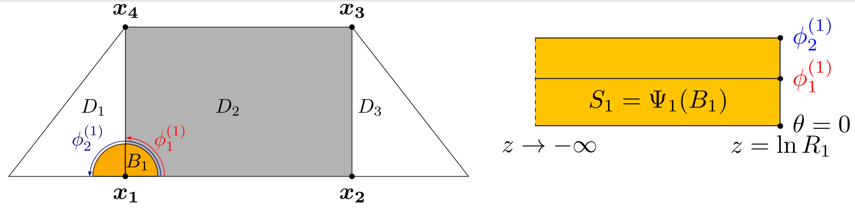


Fig. Obstacle with scaling region B_1 highlighted.

Cartesian-to-Euler coordinate mapping $\Psi_i(x, y) = (z, \theta)$:

$$\Psi_i(B_i) = S_i := (-\infty, \ln R_i) \times (0, \phi_2^{(i)}) \quad (i \in [\![1, N_c]\!]) \, .$$

Corner complex scaling: weak formulation

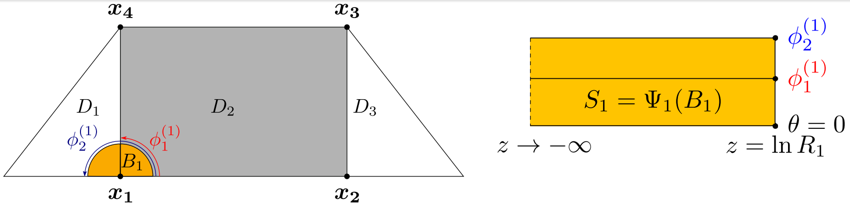


Fig. Obstacle with scaling region B_1 highlighted.

Cartesian-to-Euler coordinate mapping $\Psi_i(x,y) = (z,\theta)$:

$$\Psi_i(B_i) = S_i := (-\infty, \ln R_i) \times (0, \phi_2^{(i)}) \quad (i \in [\![1, N_c]\!]) \,.$$

Weak formulation of ITEPα is obtained with the substitutions:

$$(\nabla u, \nabla \varphi_u)_{B_i} \to \left(\nabla_i^{(\alpha)} \tilde{u}_i, \nabla \varphi_{\tilde{u}_i}\right)_{S_i} \quad \text{and} \quad (u, \varphi_u)_{B_i} \to \left(\omega_i^{(\alpha)} \tilde{u}_i, \varphi_{\tilde{u}_i}\right)_{S_i},$$

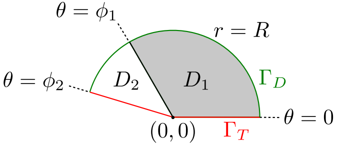
where

$$\nabla_i^{(\alpha)} u(z,\theta) := \begin{pmatrix} \alpha \partial_z u \\ \frac{1}{\alpha} \partial_\theta u \end{pmatrix}, \; \omega_i^{(\alpha)}(z,\theta) := \frac{1}{\alpha} e^{2z/\alpha}.$$

Next: validation of this weak formulation?

Corner complex scaling: validation

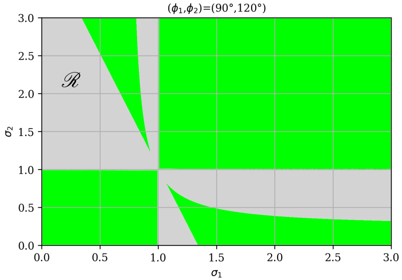
Validation obstacle



Boundary conditions:

$$u - w = 0,\ \partial_{\nu_A} u - \partial_\nu w = 0 \quad (\Gamma_T),$$
$$u = w = 0 \quad (\Gamma_D).$$

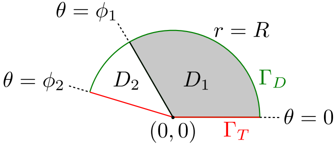
Singularity region $\mathscr{R}$



Problem: find $(\sigma_1, (u, w))$ given (σ_2, k, n). Exact spectrum: $\mathscr{R} \cap \{\sigma_2 = \text{cst}\}$.

Corner complex scaling: validation

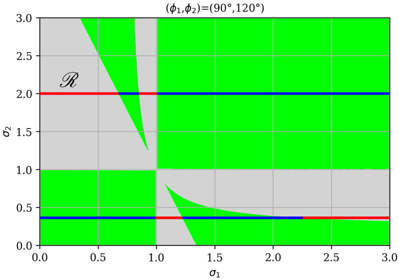
Validation obstacle



Boundary conditions:

$$u - w = 0,\ \partial_{\nu_A} u - \partial_\nu w = 0 \qquad (\Gamma_T),$$
$$u = w = 0 \qquad (\Gamma_D).$$

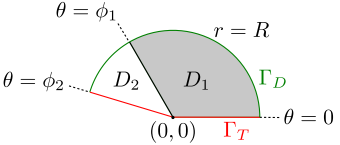
Singularity region $\mathscr{R}$



Problem: find $(\sigma_1, (u, w))$ given (σ_2, k, n). Exact spectrum: $\mathscr{R} \cap \{\sigma_2 = \text{cst}\}$.

Corner complex scaling: validation

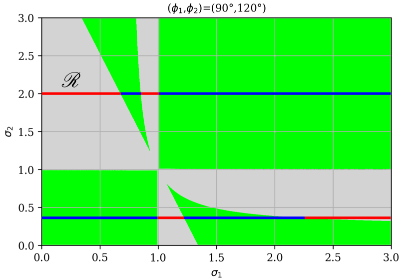
Validation obstacle



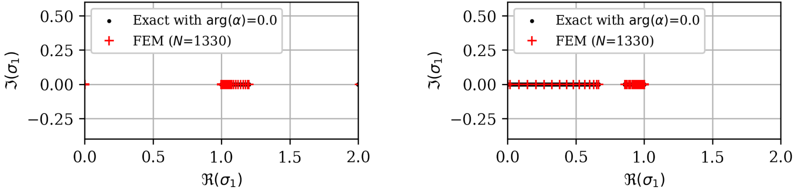
Boundary conditions:

$$u - w = 0,\ \partial_{\nu_A} u - \partial_\nu w = 0 \quad (\Gamma_T),$$
$$u = w = 0 \quad (\Gamma_D).$$

Singularity region $\mathscr{R}$

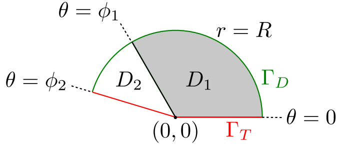


Problem: find $(\sigma_1, (u, w))$ given (σ_2, k, n). Exact spectrum: $\mathscr{R} \cap \{\sigma_2 = \text{cst}\}$.



Corner complex scaling: validation

Validation obstacle

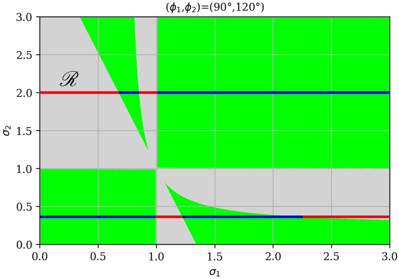


Boundary conditions:

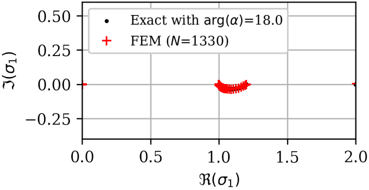
$$u - w = 0,\ \partial_{\nu_A} u - \partial_\nu w = 0 \qquad (\Gamma_T),$$

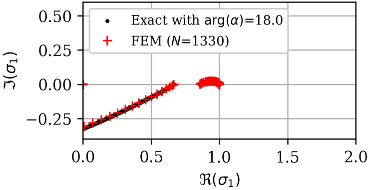
$$u = w = 0 \qquad (\Gamma_D).$$

Singularity region $\mathscr{R}$



Problem: find $(\sigma_1, (u, w))$ given (σ_2, k, n). **Exact spectrum:** $\mathscr{R} \cap \{\sigma_2 = \text{cst}\}$.





Contents

1 Introduction

2 Strongly-oscillating singularities in ITEP

3 Corner complex scaling

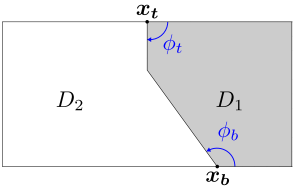
4 Numerical results

- Problem setup
- Case A
- Case B

5 Conclusion

Problem setup

Obstacle definition: $\phi_t = \pi/2$, $\phi_b = 1.4 \times \pi/2$, and $n = 1.2$

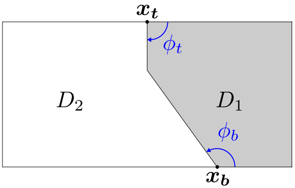


Case A: $(\sigma_1, \sigma_2) = (0.25, 0.9)$.
$\Rightarrow$ Discrete spectrum in $H^1(D)$.

Case B: $(\sigma_1, \sigma_2) = (0.25, 1.1)$.
$\Rightarrow$ Discrete spectrum in $X_\gamma(D)$.

Problem setup

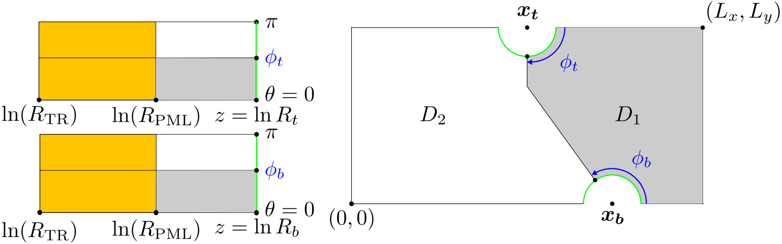
Obstacle definition: $\phi_t = \pi/2$, $\phi_b = 1.4 \times \pi/2$, and $n = 1.2$



Case A: $(\sigma_1, \sigma_2) = (0.25, 0.9)$.
$\Rightarrow$ Discrete spectrum in $H^1(D)$.

Case B: $(\sigma_1, \sigma_2) = (0.25, 1.1)$.
$\Rightarrow$ Discrete spectrum in $X_\gamma(D)$.

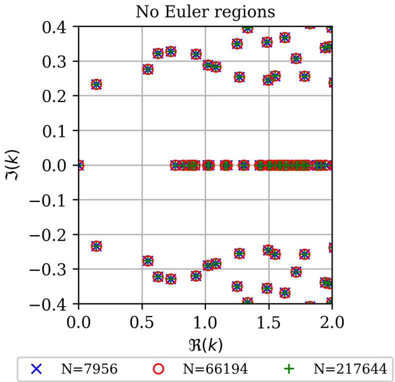
Obstacle with top and bottom scaling regions



Case A: discrete spectrum in $H^1(D)$

Convergence w.r.t. N (# of DoF) and dependency on $\alpha = e^{i\theta}$ (scaling).

Without scaling

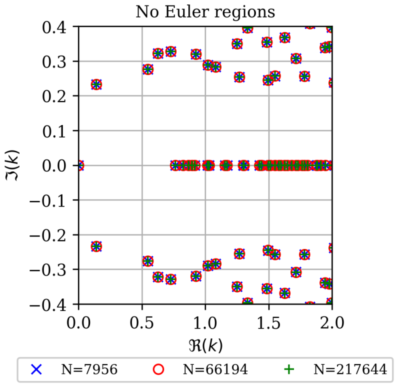


With scaling

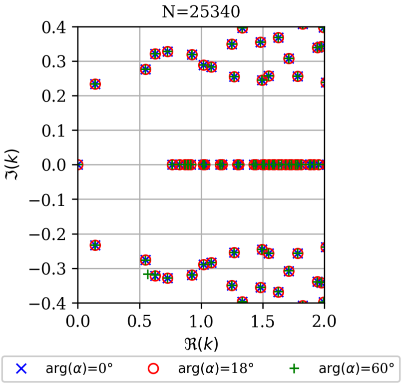
Case A: discrete spectrum in $H^1(D)$

Convergence w.r.t. N (# of DoF) and dependency on $\alpha = e^{i\theta}$ (scaling).

Without scaling

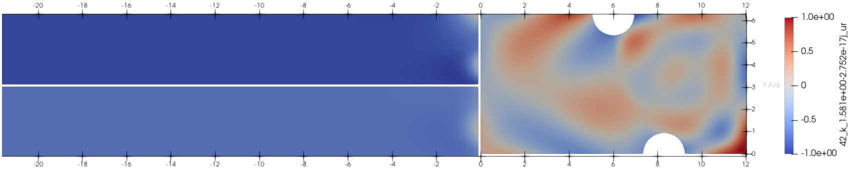


With scaling

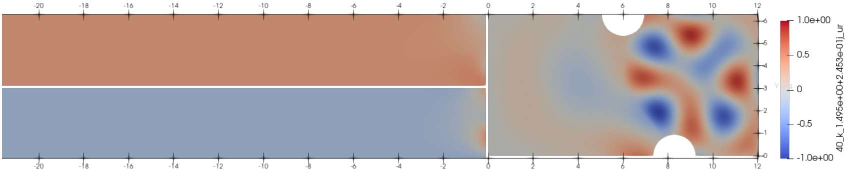


- Both weak formulations yield the same convergent spectrum.

Case A: discrete spectrum in $H^1(D)$



(a) Case A, $k \simeq 1.58$.



(b) Case A, $k \simeq 1.49 + 0.24i$.

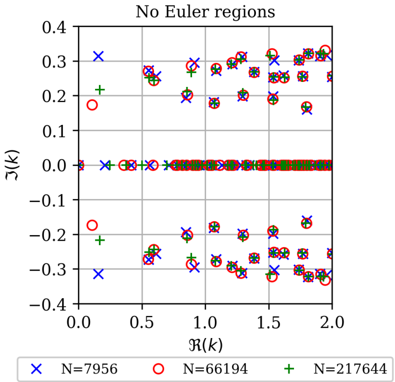
Fig. Eigenfunctions $\Re(u_h)$ computed with $\alpha = 1$ ($N = 78968$).

- No oscillations in the scaling region.

Case B: discrete spectrum in $X_\gamma(D)$

Convergence w.r.t. N (# of DoF).

Without scaling

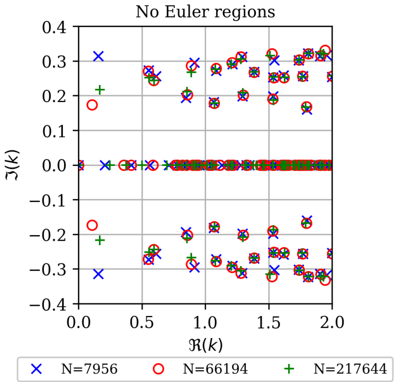


With scaling $\arg(\alpha) = \pi/10$

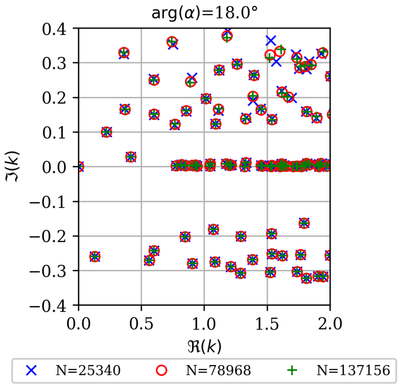
Case B: discrete spectrum in $X_\gamma(D)$

Convergence w.r.t. N (# of DoF).

Without scaling



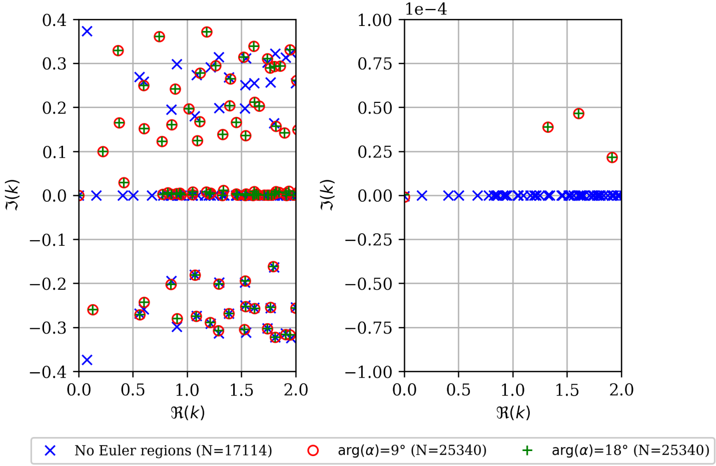
With scaling $\arg(\alpha) = \pi/10$



- Complex scaling enables to compute convergent eigenvalues

Case B: discrete spectrum in $X_\gamma(D)$

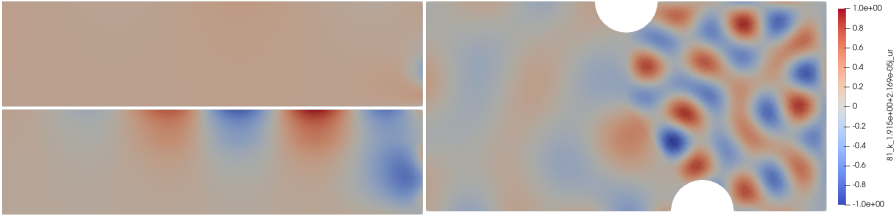
Superposition of eigenvalues with and without scaling:



- Transmission eigenvalues depends only upon of $\text{sign}(\arg(\alpha))$
- No real transmission eigenvalues

Case B: discrete spectrum in $X_\gamma(D)$

Computed eigenfunctions $\Re(u_h)$ with scaling:



Case B with $\alpha = e^{i\pi/10}$, $k \simeq 1.95 + 2.17 \cdot 10^{-5}\, i$.

Contents

1 Introduction

2 Strongly-oscillating singularities in ITEP

3 Corner complex scaling

4 Numerical results

5 Conclusion

- Conclusion and outlook

Conclusion

▸ Appendix

Code at `github.com/fmonteghetti/python-academic-projects`

Takeaways

- When $I - A$ changes sign on ∂D, it can happen that ITEP has no longer a discrete spectrum in $H^1(D)$ (loss of Fredholmess).
- Fredholmness can be restored in the alternative space $X_\gamma(D)$, with γ a free parameter. (Bonnet-Ben Dhia and Chesnel 2013)
- Corner complex scaling can be used to compute the discrete spectrum in $X_0(D)$. No real eigenvalues found so far.

Outlook

- Computation in $X_\gamma(D)$ for $\gamma \neq 0$ and finite?
- Meaning of transmission eigenvalues in $X_\gamma(D)$: Dependency on γ? Is there a physical value of γ? Link with e.g. the linear sampling method?
- Case $A = A(k)$?

Conclusion

Appendix

Code at `github.com/fmonteghetti/python-academic-projects`

Takeaways

- When $I - A$ changes sign on ∂D, it can happen that ITEP has no longer a discrete spectrum in $H^1(D)$ (loss of Fredholmess).
- Fredholmness can be restored in the alternative space $X_\gamma(D)$, with γ a free parameter. (Bonnet-Ben Dhia and Chesnel 2013)
- Corner complex scaling can be used to compute the discrete spectrum in $X_0(D)$. No real eigenvalues found so far.

Outlook

- Computation in $X_\gamma(D)$ for $\gamma \neq 0$ and finite?
- Meaning of transmission eigenvalues in $X_\gamma(D)$: Dependency on γ? Is there a physical value of γ? Link with e.g. the linear sampling method?
- Case $A = A(k)$?

Thanks for your attention.

Additional slides

◂ Conclusion

References I

- Bonnet-Ben Dhia, A.-S., C. Carvalho, L. Chesnel, and P. Ciarlet (2016). "On the use of Perfectly Matched Layers at corners for scattering problems with sign-changing coefficients". In: *Journal of Computational Physics* 322, pp. 224–247. DOI: 10.1016/j.jcp.2016.06.037 (cit. on pp. 28–30).
- Bonnet-Ben Dhia, A.-S. and L. Chesnel (Sept. 2013). "Strongly oscillating singularities for the interior transmission eigenvalue problem". In: *Inverse Problems* 29.10, p. 104004. DOI: 10.1088/0266-5611/29/10/104004 (cit. on pp. 22–24, 48, 49).
- Bonnet-Ben Dhia, A.-S., L. Chesnel, and H. Haddar (2011). "On the use of T-coercivity to study the interior transmission eigenvalue problem". In: *Comptes Rendus Mathematique* 349.11, pp. 647–651. DOI: 10.1016/j.crma.2011.05.008 (cit. on pp. 8–12).
- Cakoni, F., D. Colton, and H. Haddar (Oct. 2021). "Transmission Eigenvalues". In: *Notices of the American Mathematical Society* 68.09, pp. 1499–1510. DOI: 10.1090/noti2350 (cit. on pp. 5–7).

References II

Cakoni, F. and H. Haddar (2012). "Transmission Eigenvalues in Inverse Scattering Theory". In: *Inside Out II*. Ed. by G. Uhlmann. Vol. 60. (hal-00741615). MSRI Publications, pp. 527–578 (cit. on pp. 5–7).