Optimized Schwarz Method in Time Direction for Transport Control

Duc Quang Bui, Bérangère Delourme, Laurence Halpern et Félix Kwok

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Motivation

We are interested in optimization problems governed by hyperbolic partial differential equations (PDEs) from inverse problem/data assimilation.

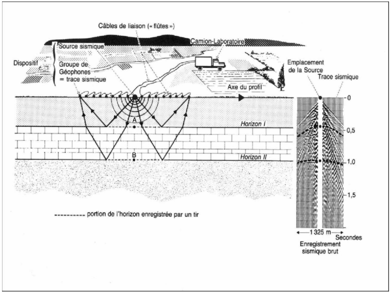


Figure: Expérience de sismique réflexion.

- Model : $\dot{x}_{\rm ref}(t) = Ax_{\rm ref}(t) + Bv(t)$.
- Linear data : $y_{\rm ref}(t) = Cx_{\rm ref}(t)$.
- Objective function :

$$J(u) = \frac{1}{2} F_1(\|Cx_{\rm ref} - y_{\rm obs}\|) + F_2(\|v\|)$$

Difficulties

- large amounts of data to process
- high resolution required.
- backward-Forward optimization loops.

$\Rightarrow$ essential to design scalable, highly efficient parallelel methods.

Domain Decomposition Methods

Replace solving a PDE in a large/complex domain by solving successively the same PDE in the smaller/simpler subdmains

For PDEs

- Overlapping Schwarz Method
  · · · [Schwarz 1870]
- Non-overlapping + Robin conditions
  · · · Lions (1990).
- Optimized conditions
  · · · [Japhet-Nataf 2001], [Gander 2006].

For Optimal Control Problem

- Schwarz Method for eliptic optimal control
  · · · [Benamou 1994], [Benamou-Després 1996]

For *time-dependent* PDEs

- Parareal
  · · · [Lions-Maday-Turinici 01], [Gander-Vandewalle 07]
- Space-Time DD
  · · · [Gander-Halpern-Nataf 99], [Gander-Kwok-Mandal 16]

For *Time* Optimal Control Problem

- PinT in optimization loops
  · · ·[Götschel-Minion 19], [Günther and al. 19]
- ParaOpt · · · [Gander-Kwok-Salomon 20]
- Domain decomposition in Time direction.
  · · · [Gander-Kwok 16], [Leugering and al. 21]

Discription of the problem

Let $T > 0$, and $y_{\text{ini}}, y_{\text{tar}} \in L^2(\mathbb{R})$.
We find a control $v \in L^2(0, T, L^2(\mathbb{R}))$ s.t. y defined by

$$\begin{cases} \partial_t y + \partial_x y = v & \text{in } \mathbb{R} \times (0, T), \\ y(., 0) = y_{\text{ini}}, \end{cases}$$

verifies the exact constraint

$$y(., T) = y_{\text{tar}}.$$

We shall seek v that minimize the functional

$$J(v) = \frac{1}{2} \int_0^T \|v\|^2_{L^2(\mathbb{R})}.$$

The above optimization problem has a unique solution $v^* \in L^2(\mathbb{R} \times (0, T))$, given by

$$v^* = \lambda,$$

where,

$$\begin{cases} \partial_t y + \partial_x y = \lambda \\ \partial_t \lambda + \partial_x \lambda = 0 \end{cases} \qquad y(t = 0) = y_{\text{ini}}\,, y(t = T) = y_{\text{tar}} \tag{1}$$

Equivalent PDEs in y

$$\partial_{tt} y + 2\partial_{tx} y + \partial_{xx} y = 0$$

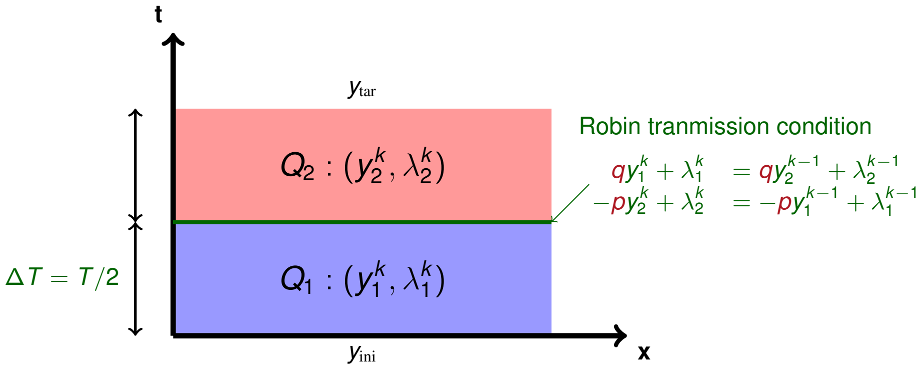
with 2-point boundary at $t = 0$ and $t = T \Rightarrow$ Schwarz Method.

OUTLINE

Time Direction Schwarz Algorithm

Starting with (y_1^0, λ_1^0) and (y_2^0, λ_2^0), at iteration $k \geq 1$, we solve

$$\begin{cases} \partial_t y_1^k + \partial_x y_1^k &= \lambda_1^k \\ \partial_t \lambda_1^k + \partial_x \lambda_1^k &= 0 \end{cases} \qquad \begin{cases} \partial_t y_2^k + \partial_x y_2^k &= \lambda_2^k \\ \partial_t \lambda_2^k + \partial_x \lambda_2^k &= 0 \end{cases}$$



Subdomain system is equivalent with an optimal control problem

· · · [Leugeuring and al. 2021]

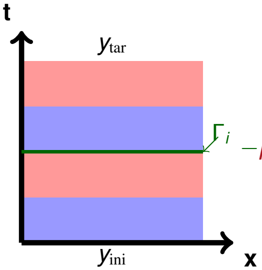
Convergence

Theorem

The optimal choice is $p = \frac{1}{\Delta T}, q = \frac{1}{\Delta T}$, *which leads to an immediate convergence after* 2 *iterations.*

Remark

For general N time windows, with $p_i = \frac{1}{2^{i-1}\Delta T}$ and $q_i = \frac{1}{2^{N-i-1}\Delta T}$, the method converges after N iterations.



$$
\begin{aligned}
q_i y_i^k + \lambda_i^k &= q y_{i+1}^{k-1} + \lambda_{i+1}^{k-1} \\
-p_i y_{i+1}^k + \lambda_{i+1}^k &= -p_i y_i^{k-1} + \lambda_i^{k-1}
\end{aligned}
$$

Proof: Fourrier analysis

Using Fourier transform in space :

For $\hat{y}_{\text{ini}}, \hat{y}_{\text{tar}} \in \mathbb{C}$, we minimize

$$\min_{\hat{v} \in L^2(0,T)} J(\hat{v}) = \frac{1}{2} \int_0^T |\hat{v}|^2$$

where the control $\hat{v}$ is such that $\hat{y}$

$$\begin{cases} \partial_t \hat{y} + \mathrm{i}\omega \hat{y} = \hat{v} \quad t \in (0, T), \\ \hat{y}(0) = \hat{y}_{\text{ini}}, \end{cases}$$

satisfies

$$\hat{y}(T) = \hat{y}_{\text{tar}}$$

The unique solution $\hat{v}^* \in L^2(0, T)$ is given by

$$\hat{v}^* = \hat{\lambda},$$

where,

$$\begin{cases} \partial_t \hat{y} + \mathrm{i}\omega \hat{y} = \hat{\lambda} \\ \partial_t \hat{\lambda} + \mathrm{i}\omega \hat{\lambda} = 0 \end{cases} \qquad \hat{y}(0) = \hat{y}_{\text{ini}}, \hat{y}(T) = \hat{y}_{\text{tar}} \tag{2}$$

Fourier Analysis setting: Convergence Factor

Let $\hat{y}_{\text{ini}} = \hat{y}_{\text{tar}} = 0$. Denote by $\hat{\xi}_i^k$ the Robin terms.

Starting from $\hat{\xi}_1^0$ and $\hat{\xi}_2^0$, at iteration $k \geq 1$, we solve (2) in $(0, \Delta T)$ and $(\Delta T, T)$ with

$$\begin{aligned} q\hat{y}_1^k + \hat{\lambda}_1^k &= \xi_1^{k-1} \\ -p\hat{y}_2^k + \hat{\lambda}_2^k &= \xi_2^{k-1} \end{aligned} \qquad \text{then update} \qquad \begin{aligned} \hat{\xi}_1^k &= q\hat{y}_2^k + \hat{\lambda}_2^k \\ \hat{\xi}_2^k &= -p\hat{y}_1^k + \hat{\lambda}_1^k \end{aligned}$$

Convergence factor is the $\sqrt{|\rho|}$ defined by

$$\hat{\xi}_i^k = \rho \hat{\xi}_i^{k-2}$$

$$\rho := \rho(p, q) = \frac{1 - p\Delta T}{1 + p\Delta T} \cdot \frac{1 - q\Delta T}{1 + q\Delta T}$$

OUTLINE

1. Schwarz Method for continuous settings
2. Schwarz Method for discrete model
3. Asymptotic study of the convergence factor
4. Numerical Illustrations
5. Conclusion and Perspective

Fully discrete optimal control problem

We consider y and v periodic in space of periodic one.

$$\min_{v^1,\dots,v^M \in \mathbb{R}^{N_x}} J(v^1,\dots,v^M) = \Delta x \Delta t \frac{1}{2} \sum_{m=1}^{M} \|v^m\|^2_{\mathbb{R}^{N_x}}$$

where the control $(v^1,\dots,v^M)$ is s.t. $(y_0,\dots,y_M)$ verifies, for $m = 1,\dots,M$

$$\begin{cases} \dfrac{y_j^m - y_j^{m-1}}{\Delta t} + \dfrac{y_j^{m-1} - y_{j-1}^{m-1}}{\Delta x} = v_j^m, \\ y_j^0 = y_{\text{ini},j}, \end{cases}$$

satisfies $y_j^M = y_{\text{tar},j}$.

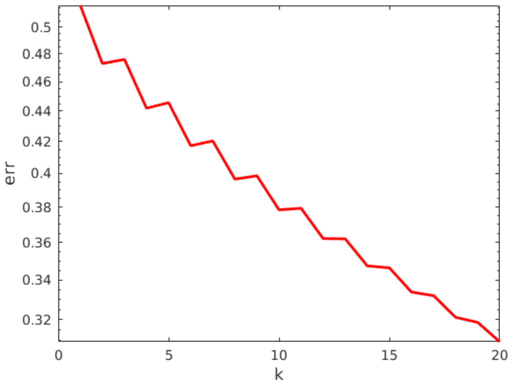
Remark

At $t_1 = \Delta t$, for $y_{\text{ini}} = 0$, $y(.,\Delta t) = \Delta t v(.,\Delta t) \Rightarrow v_j^{m-1} \to v_j^m$.

$$\begin{cases} \dfrac{y_j^m - y_j^{m-1}}{\Delta t} + \dfrac{y_j^{m-1} - y_{j-1}^{m-1}}{\Delta x} = \lambda_j^m, \\ \dfrac{\lambda_j^{m-1} - \lambda_j^m}{\Delta t} + \dfrac{\lambda_j^m - \lambda_{j+1}^m}{\Delta x} = 0, \\ y_j^0 = y_{\text{ini},j}, y_j^M = y_{\text{tar},j} \end{cases}$$

Convergence

$T = 1$, $\Omega = (0, 1)$, $\Delta t = 1/160$, $r = 1/2$;



$\Rightarrow$ We do not have convergence after 2 iterations!

Discrete Fourier Analysis

Discrete Fourier Transform

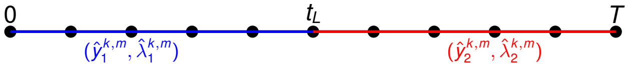
$$\frac{y_j^{m-1} - y_{j-1}^{m-1}}{\Delta x} \to \sigma(\ell)\hat{y}^m, \quad \frac{\lambda_j^m - \lambda_{j+1}^m}{\Delta x} \to \bar{\sigma}(\ell)\hat{\lambda}^m$$

with $\sigma(\ell) = \dfrac{1 - e^{-2\pi \mathrm{i}\ell\Delta x}}{\Delta x}$, $\ell = 0, \ldots, N_x - 1$.

$$\begin{cases} \dfrac{\hat{y}^m - \hat{y}^{m-1}}{\Delta t} + \sigma(\ell)\hat{y}^{m-1} = \hat{\lambda}^m \\ \dfrac{\hat{\lambda}^m - \hat{\lambda}^{m-1}}{\Delta t} - \bar{\sigma}\hat{\lambda}^m = 0 \\ y^0 = \hat{y}_{\mathrm{ini}} \\ y^M = \hat{y}_{\mathrm{tar}} \end{cases} \tag{3}$$

$\dfrac{\Delta t}{\Delta x} = r$ fixed $\to$ maximum of ℓ is of order $(\Delta t)^{-1}$.

Convergence factor



We solve (3) in $(0, \Delta T)$ $(m = 1, \dots, M/2)$ and in $(\Delta T, T)$ $(m = M/2 + 1, \dots, M)$ with the transmission conditions at the interface $m = M/2$

$$q\hat{y}_1^{k,M/2} + \hat{\lambda}_1^{k,M/2} = \hat{\xi}_1^{k-1} \qquad -p\hat{y}_2^{k,M/2} + \hat{\lambda}_2^{k,M/2} = \hat{\xi}_2^{k-1}$$

Then update

$$\hat{\xi}_1^{k} = q\hat{y}_2^{k,M/2} + \hat{\lambda}_2^{k,M/2} \qquad \hat{\xi}_2^{k} = -p\hat{y}_1^{k,M/2} + \hat{\lambda}_1^{k,M/2}$$

We get $\hat{\xi}_i^{k} = \rho\hat{\xi}_i^{k-2}$ with

$$\rho := \rho_{\Delta t}(p, q, \ell) = \frac{1 - p\gamma_{\Delta t}(\ell)}{1 + q\gamma_{\Delta t}(\ell)} \cdot \frac{|\beta_{\Delta t}(\ell)|^2 - q\gamma_{\Delta t}(\ell)}{|\beta_{\Delta t}(\ell)|^2 + p\gamma_{\Delta t}(\ell)}$$

where $\beta_{\Delta t}(\ell) = (1 - \sigma(\ell)\Delta t)^{M/2}$, and $\gamma_{\Delta t}(\ell) = \Delta t \sum_{m=0}^{M/2-1} |1 - \sigma(\ell)\Delta t|^{2m}$.

Minimax Propblem

The convergence factor depends on ℓ and on Δt (Δx fixed by Δt), denoted by $\rho_{\Delta t}$

$$\min_{p,q>0} \max_{\ell=0,\dots,N_x-1} |\rho_{\Delta t}(p,q,\ell)| \tag{4}$$

$\rho_{\Delta t}$ is a function in

$$|1-\Delta t\sigma(\ell)|^2 = 1-4r(1-r)\sin^2(\pi\ell\Delta x)$$

$\Rightarrow$ new variable $s = 4r(1-r)\sin^2(\pi\omega\Delta x)$, $s \in [0, s_{\max}]$ with $s_{\max} = 4r(1-r)$.
Optimial parameters $\dfrac{1}{\Delta T} \Rightarrow p\Delta T$ and $q\Delta T$ as parameters.
New minimax problem

$$\min_{p,q>0} \max_{0\leq s\leq s_{\max}} |\rho_{\Delta t}(p,q,s)| \tag{5}$$

OUTLINE

One parameter case $p = q$

$\rho_{\Delta t}(p, s) := \rho_{\Delta t}(p, p, s)$

$$\min_{p>0} \max_{0 \leq s \leq s_{\max}} |\rho_{\Delta t}(p, s)| \tag{6}$$

Theorem

For Δt small enough, Problem (6) has a unique solution $p^*_{\Delta t}$. As Δt goes to 0,

$$p^*_{\Delta t} = \sqrt{2\Delta T\, s_{\max}}.\Delta t^{-1/2} + o\left(\Delta t^{-1/2}\right),$$

$$\max_{0 \leq s \leq s_{\max}} |\rho_{\Delta t}(s, p^*_{\Delta t})| = 1 - \frac{2\sqrt{2}}{\sqrt{\Delta T\, s_{\max}}} \cdot \Delta t^{1/2} + o\left(\Delta t^{1/2}\right).$$

Sketch of the proof

Step 1 : Solution of alternation equation

Let $p_{\max} = \dfrac{\Delta T}{\gamma_{\Delta t}(s_{\max})}$.

$$\max_{0 \le s \le s_{\max}} \rho_{\Delta t}(p, z) = - \min_{0 \le z \le s_{\max}} \rho_{\Delta t}(p, s).$$

has a unique solution $1 < p^*_{\Delta t} < p_{\max}$. We introduce

$$\rho_{\Delta t,\max}(p) = \max_{0 \le s \le s_{\max}} \rho_{\Delta t}(p, z), \quad \rho_{\Delta t,\min}(p) = \min_{0 \le s \le s_{\max}} \rho_{\Delta t}(p, z),$$

and the function

$$f = \rho_{\Delta t,\max} + \rho_{\Delta t,\min}.$$

For $p > 1$,

$$\frac{\partial \rho_{\Delta t}}{\partial p}(p, z) > 0.$$

.

$\Rightarrow$ $\rho_{\Delta t,\max}$, $\rho_{\Delta t,\min}$ and f are strictly increasing.
$f(1) < 0$, $f(p_{\max}) > 0 \Rightarrow$ unique $p^*_{\Delta t}$.

Sketch of the proof

Step 2 : $p^*_{\Delta t}$ is the unique solution of the minimax problem

We can prove that (6) has solutions, which must be in $(1, p_{\max})$.

- For $p \in (1, p^*_{\Delta t})$, a careful investigation leads to

$$\max_{0 \leq s \leq s_{\max}} |\rho_{\Delta t}(p, s)| = -\rho_{\Delta t,\min}(p) > -\rho_{\Delta t,\min}(p^*_{\Delta t}) = \max_{0 \leq s \leq s_{\max}} |\rho_{\Delta t}(p^*_{\Delta t}, s)|.$$

- Similarly, for $p \in (p^*_{\Delta t}, p_{\max})$, we obtain

$$\max_{0 \leq z \leq s_{\max}} |\rho_{\Delta t}(p, s)| = \rho_{\Delta t,\max}(p) > \rho_{\Delta t,\max}(p^*_{\Delta t}) = \max_{0 \leq s \leq s_{\max}} |\rho_{\Delta t}(p^*_{\Delta t}, s)|.$$

Therefore, $p^*_{\Delta t}$ is the unique global minimum of (6).

Sketch of the proof

Step 3 : Asymptotics of $p^*_{\Delta t}$ and $\max_{0\leq s\leq s_{\max}} |\rho_{\Delta t}(p^*_{\Delta t}, s)|$ with respect to Δt

We approximate $\rho_{\Delta t,\max}(p)$ by $\rho_{\Delta t}(p, 0)$ using

$$|\rho_{\Delta t,\max}(p) - \rho_{\Delta t}(p, 0)| \leq Cp^{-1}\Delta t.$$

$\Rightarrow$ 'approximate' equation

$$\rho_{\Delta t}(p, 0) = -\rho_{\Delta t}(p, s_{\max}),$$

whose solution $p^*_{\mathrm{eq},\Delta t}$ can be calculated explicitly and has asymptotic

$$p^*_{\mathrm{eq},\Delta t} = \sqrt{2\Delta T\, s_{\max}}.\Delta t^{-1/2} + o\left(\Delta t^{-1/2}\right),$$

which implies

$$-\rho_{\Delta t}(p^*_{\mathrm{eq},\Delta t}, s_{\max}) = \rho_{\Delta t}(p^*_{\mathrm{eq},\Delta t}, 0) = 1 - \frac{2\sqrt{2}}{\sqrt{\Delta T\, s_{\max}}} \cdot \Delta t^{1/2} + o\left(\Delta t^{1/2}\right).$$

Two parameters case

Theorem

For Δt small enough, Problem (6) *has a unique solution* $(p^*_{\Delta t}, q^*_{\Delta t})$. *When* $\Delta t \to 0$

$$p^*_{\Delta t} = p^* + O(\Delta t)\,, \quad q^*_{\Delta t} = q^* + O(\Delta t)$$

$$\max_{0 \le s \le s_{\max}} |\rho_{\Delta t}(p^*_{\Delta t}, q^*_{\Delta t}, s)| = \rho^* + O(\Delta t)\,.$$

where $p^* \simeq 1.1993$, $q^* \simeq 0.0906$ *and* $\rho^* \simeq 0.0755$.

Remark

- $(p^*_{\Delta t}, q^*_{\Delta t})$ and $\rho_{\Delta t}$ do not converge to corresponding continuous values.
- Similar results as in the parabolic case
  ··· [Gander-Kwok 16]

Sketch of the proof

Step 1 : Minimax problem (5) has solutions $(p^*_{\Delta t}, q^*_{\Delta t})$ with $1 \leq p^*_{\Delta t} \leq p_{\max}$. In additions, the solution must verify

$$\rho_{\Delta t}(p, q, s = 0) = \rho_{\Delta t}(p, q, s = s_{\max}).$$

which implies $q = q_{\Delta t,\mathrm{eq}}(p)$. Denote by $\rho_{\Delta t,\mathrm{eq}}(p, s) = \rho_{\Delta t}(p, q_{\Delta t,\mathrm{eq}}(p), s)$. We solve

$$\min_{1 \leq p \leq p_{\max}} \max_{0 \leq s \leq s_{\max}} |\rho_{\Delta t,\mathrm{eq}}(p, s)|$$

Step 2 : The minimax solution must verify the equation

$$-\rho_{\Delta t,\mathrm{eq}}(p, s = 0) = \max_{0 \leq s \leq s_{\max}} \rho_{\Delta t,\mathrm{eq}}(p, s)$$

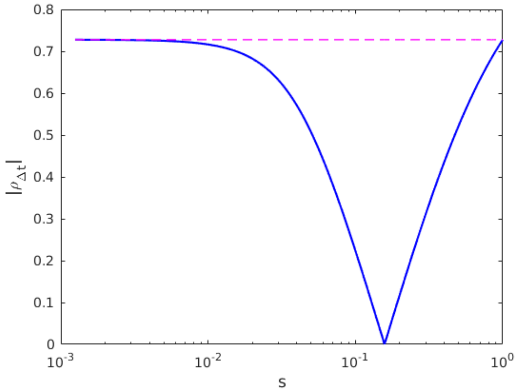
Step 3 : We calculate the limit version of Equiocisllation system for $\Delta t \to 0$.

OUTLINE

Optimized Convergence factor with one paramter

$p^*_{\Delta t}$ and $\max_{0 \leq s \leq s_{\max}} |\rho_{\Delta t}(p^*_{\Delta t}, s)|$ calculated numerically

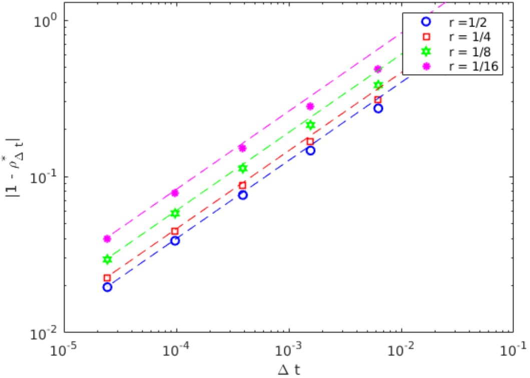
- For $\Delta t = 1/160$: Equiocisllation and $\max_{0 \leq s \leq s_{\max}} \rho_{\Delta t}(p^*_{\Delta t}, s) \simeq \rho_{\Delta t}(p^*_{\Delta t}, 0)$.



Optimized Convergence factor with one paramter

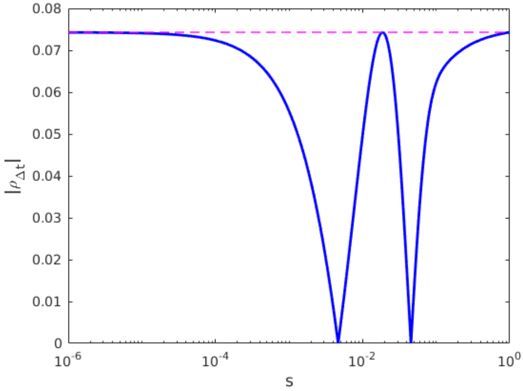
$p^*_{\Delta t}$ and $\max_{0 \le s \le s_{\max}} |\rho_{\Delta t}(p^*_{\Delta t}, s)|$ calculated numerically

- $r = \dfrac{\Delta t}{\Delta x}$ in different colors.
- Markers : $\left|1 - \max_{0 \le s \le s_{\max}} |\rho_{\Delta t}(p^*_{\Delta t}, s)|\right|$. Dash lines : $\dfrac{2\sqrt{2}}{\sqrt{\Delta T\ s_{\max}}} \cdot \Delta t^{1/2}$.



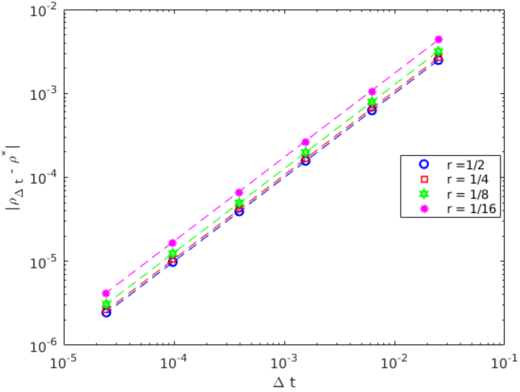
Optimized Convergence factor with two paramters

- For $\Delta t = 1/160$: Triple-Equiocisllation at $(p^*_{\Delta t}, q^*_{\Delta t})$.



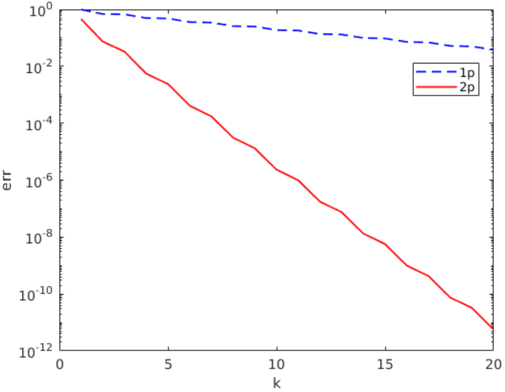
Optimized Convergence factor with two paramters

- Markers : $\left|\rho^* - \max_{0 \le s \le s_{\max}} |\rho_{\Delta t}(p^*_{\Delta t}, q^*_{\Delta t}, s)|\right|$. Dash lines: best fit line $C\Delta t$



Pratical performance

$T = 1$, $r = 1/2$, $\Delta t = 1/160$, $y_{\text{ini}} = y_{\text{tar}} = 0$, random initial Robin terms.



OUTLINE

Conclusion and Perspective

Conlusion

- Asymptotic Analysis in case of Explicit Euler - Upwind scheme.
- Numerical illustration for time-dependent advection.

Ongoing work

- Convergence Analysis for other scheme (e.g. Lax-Wendroff).
- Extension to multi-domain.
- Para-Opt for Optimal transport Control.

Thank you for your attention