Research on the inverse acoustic scattering problems in inhomogeneous media

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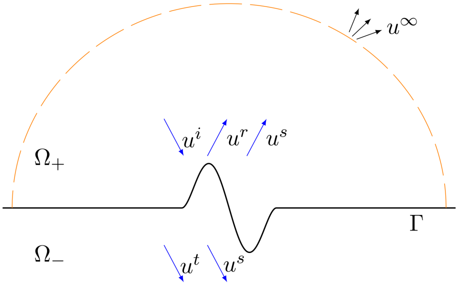
1. Scattering by a locally rough surface

▶ Consider scattering of time-harmonic acoustic waves by a penetrable locally perturbed infinite plane (called a locally rough surface).

▶ This type of problems occurs in various areas of applications such as: radar and sonar detection, remote sensing, diffractive optics and nondestructive testing.

▶ Let $\Gamma := \{(x_1, x_2) : x_2 = h_\Gamma(x_1), x_1 \in \mathbb{R}\}$ represent a locally rough surface, where $h_\Gamma \in C^2(\mathbb{R})$ has a compact support in $\mathbb{R}$.

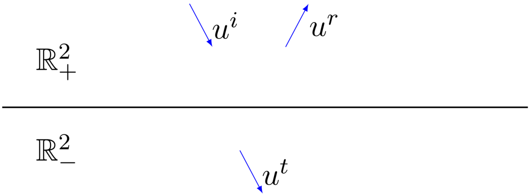


1. Scattering by a locally rough surface

- Let $k_\pm > 0$ be two different wave numbers in $\Omega_\pm$, respectively.
- Let $d = (\cos\theta_d, \sin\theta_d)$ with $\theta_d \in (\pi, 2\pi)$.
- Given an incident plane wave $u^i(x,d) := e^{ik_+ x\cdot d}$, then the reference wave $u^0(x,d)$ is generated by the incident field $u^i(x,d)$ and the unperturbed two-layered medium, and is given by

$$u^0(x,d) := \begin{cases} u^i(x,d) + u^r(x,d), & x \in \mathbb{R}^2_+, \\ u^t(x,d), & x \in \mathbb{R}^2_-, \end{cases}$$

where $\mathbb{R}^2_\pm := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \gtrless 0\}$.



1. Scattering by a locally rough surface

- Let $d^r := (\cos\theta_d, -\sin\theta_d)$ and $d^t := n^{-1}(\cos\theta_d, -i\mathcal{S}(\cos\theta_d, n))$ with $n = k_-/k_+$.
- From the Fresnel formula, the reflected wave $u^r(x,d)$ and transmitted wave $u^t(x,d)$ are given by

$$u^r(x,d) := \mathcal{R}(\pi+\theta_d)e^{ik_+x\cdot d^r}, \quad u^t(x,d) := \mathcal{T}(\pi+\theta_d)e^{ik_-x\cdot d^t},$$

where

$$\mathcal{R}(\theta) := \frac{i\sin\theta + \mathcal{S}(\cos\theta, n)}{i\sin\theta - \mathcal{S}(\cos\theta, n)}, \quad \mathcal{T}(\theta) := \mathcal{R}(\theta) + 1 \quad \text{for } \theta \in \mathbb{R},$$

with

$$\mathcal{S}(\cos\theta, n) = \begin{cases} -i\sqrt{n^2 - \cos\theta^2} & \text{if } |\cos\theta| \le n, \\ \sqrt{\cos\theta^2 - n^2} & \text{if } |\cos\theta| > n. \end{cases}$$

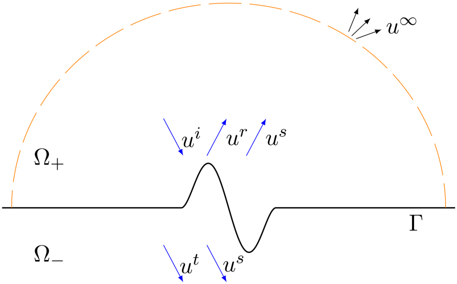
- If $|\cos\theta_d| \le n$, then $d^t = (\cos\theta_d^t, \sin\theta_d^t)$ with $\theta_d^t \in [\pi, 2\pi]$ satisfying $\cos\theta_d^t = n^{-1}\cos\theta_d$.

1. Scattering by a locally rough surface

▶ Direct scattering problem: find the total field $u^{tot} = u^0 + u^s$ such that the total field u^{tot} and the scattered field u^s satisfy

$$\begin{aligned} \Delta u^{tot} + k_\pm^2 u^{tot} &= 0 \quad \text{in} \quad \Omega_\pm, \\ [u^{tot}] = 0, \ \left[\partial u^{tot}/\partial \nu\right] &= 0 \quad \text{on} \quad \Gamma, \\ \lim_{|x|\to+\infty} \sqrt{|x|} \left(\frac{\partial u^s}{\partial |x|} - ik_\pm u^s \right) &= 0 \quad \text{uniformly for all } \hat{x} = x/|x| \in \mathbb{S}_\pm^1, \end{aligned}$$

where $[\cdot]$ denotes the jump across the interface Γ.



Inverse scattering by a locally rough surface

- Asymptotic behavior of the scattered wave u^s [1]

$$u^s(x,d) = \frac{e^{ik_+|x|}}{\sqrt{|x|}} u^\infty(\widehat{x},d) + o\left(\frac{1}{\sqrt{|x|}}\right), \quad |x| \to \infty, \quad x \in \Omega_+,$$

where $u^\infty(\widehat{x},d)$ is called the far-field pattern of the scattered field $u^s(x,d)$.

- The inverse scattering problem with phased data

Given the scattered field $u^s(x,d)$ or the far-field pattern $u^\infty(\widehat{x},d)$, determine the locally rough interface Γ.

- Existing numerical algorithms with phased scattered-field data

Li-Sun-Zhang' 17, Liu-Zhang-Zhang' 18, Zhang' 20, Li-Yang-Zhang' 21, Li-Yang' 22

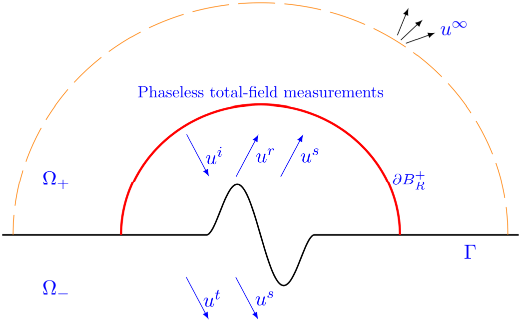
[1] H. Ammari, E. Iakovleva and D. Lesselier, *Multiscale Model. Simul.* **3** (2005), 597–628.

Imaging of interfaces with phaseless total-field data

- However, in many practical applications, only the intensity (or modulus) of the field (phaseless data) is available.
- Given the phaseless total-field data

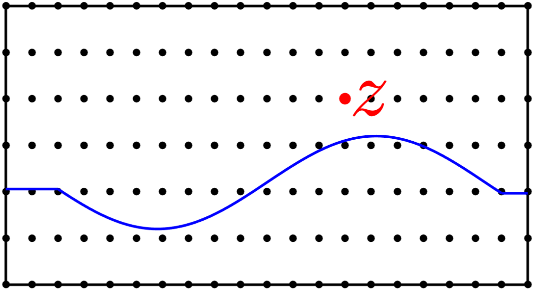
$$|u(x,d)| = |u^i(x,d) + u^r(x,d) + u^s(x,d)|,\ x \in \partial B_R^+,\ d \in \mathbb{S}^1_-,$$

reconstruct the locally rough surface Γ.



Direct imaging method for recovering the interface

Construct an indicator function $I(z)$ directly from the data s.t. the indicator function has a large contrast at the boundary and decays as z moves away from the boundary.



Direct imaging method for recovering the interface

Indicator function:

$$I_P(z,R) := \int_{\partial B_R^+} \left| \int_{\mathbb{S}^1_-} \left\{ \left[|u^{tot}(x,d)|^2 - \left(1 + |\mathcal{R}(\theta_d+\pi)|^2 + \overline{\mathcal{R}(\theta_d+\pi)} e^{2ik_+ x_2 d_2}\right)\right] e^{ik_+(x-z)\cdot d} - e^{ik_+(x'-z')\cdot d} \right\} ds(d) \right|^2 ds(x)$$

Remark: $I_P(z,R)$ is an oscillatory integral if R is large.

Direct imaging method for recovering the interface

Indicator function: $I_P(z,R) := I_S(z,R) + I_{P,Res}(z,R)$, where

$$I_S(z,R) := \int_{\partial B_R^+} |U(x,z)|\, ds(x) \quad \text{with}$$

$$U(x,z) := \int_{\mathbb{S}^1_-} u^s(x,d) e^{-ik_+ z\cdot d} + \mathcal{R}(\theta_d + \pi) e^{ik_+(x'-z)\cdot d} - e^{ik_+(x'-z')\cdot d} ds(d).$$

We need to analyse the following properties.

- The properties of $I_S(z,R)$ [2] [3]
- The asymptotic properties of $I_{P,Res}(z,R)$ as $R \to +\infty$
  - Theory of oscillatory integrals
  - Uniform far-field asymptotics of the scattered wave $u^s(x,d)$ as $|x| \to +\infty$

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[2] X. Liu, B. Zhang and H. Zhang, SIAM J. Imaging Sci. 11 (2018), 1629-1650.

[3] Hai-wen Zhang, Acta Mathematicae Applicatae Sinica, English Series 36 (2020), 119-133.

Lemma 1.1 (Z. Chen and G. Huang' 17)

For any $-\infty < a < b < \infty$, *let* $u \in C^2[a,b]$ *be real-valued and satisfy that* $|u'(t)| \geq 1$ *for all* $t \in (a,b)$. *Assume that* $a = x_0 < x_1 < \cdots < x_N = b$ *is a division of* (a,b) *such that* u' *is monotone in each interval* (x_{i-1}, x_i), $i = 1, \ldots, N$. *Then for any function* ϕ *defined on* (a,b) *with integrable derivative and for any* $\lambda > 0$,

$$\left| \int_a^b e^{i\lambda u(t)} \phi(t) dt \right| \leq (2N+2)\lambda^{-1} \left[|\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

Far-field asymptotics of the scattered field

Theorem 1 (L. Li, J. Yang, B. Zhang and H. Zhang' 22)

Assume that $k_+ < k_-$. Then $u^s(x, d)$ has the asymptotic behavior

$$u^s(x, d) = \frac{e^{ik_+|x|}}{\sqrt{|x|}} u^\infty(\hat{x}, d) + O(|x|^{-3/2}),$$

uniformly for all $\theta_{\hat{x}} \in (0, \pi)$ and $d \in \mathbb{S}^1_-$.

Far-field asymptotics of the scattered field

Theorem 2 (L. Li, J. Yang, B. Zhang and H. Zhang' 22)

Assume $k_+ > k_-$ *and* $\theta_c = \arccos(k_-/k_+)$. *Then* $u^s(x,d)$ *has the asymptotic behaviors*

$$u^s(x,d) = \frac{e^{ik_+|x|}}{\sqrt{|x|}} u^\infty(\hat{x},d) + u^s_{Res}(x,d),$$

where

- $u^s_{Res}(x,d) = O(|x|^{-3/4}), \quad |x| \to +\infty$ *uniformly for all* $\theta_{\hat{x}} \in (0,\pi)$ *and* $d \in \mathbb{S}^1_-$,
- $u^s_{Res}(x,d) = O(|\theta_c - \theta_{\hat{x}}|^{-\frac{3}{2}} |x|^{-\frac{3}{2}}), \quad |x| \to +\infty$ *uniformly for all* $\theta_{\hat{x}} \in (0,\theta_c) \cup (\theta_c, \pi/2)$ *and* $d \in \mathbb{S}^1_-$,
- $u^s_{Res}(x,d) = O(|\pi - \theta_c - \theta_{\hat{x}}|^{-\frac{3}{2}} |x|^{-\frac{3}{2}}), \quad |x| \to +\infty$ *uniformly for all* $\theta_{\hat{x}} \in [\pi/2, \pi - \theta_c) \cup (\pi - \theta_c, \pi)$ *and* $d \in \mathbb{S}^1_-$.

Direct imaging method for recovering the interface

$I_F(z) :=$

$$\int_{\mathbb{S}^1_+} \left| \int_{\mathbb{S}^1_-} u^\infty(\hat{x}, d) e^{-ik_+ z\cdot d} ds(d) + \left(\frac{2\pi}{k_+}\right)^{\frac{1}{2}} e^{-\frac{i\pi}{4}} \left(\mathcal{R}(\theta_{\hat{x}}) e^{-ik_+ \hat{x}\cdot z'} - e^{-ik_+ \hat{x}\cdot z}\right) \right|^2 ds(\hat{x})$$

Theorem 3 (L. Li, J. Yang, B. Zhang and H. Zhang' 23)

For $z \in \mathbb{R}^2$ and $R > 0$ large enough, we have $I_P(z,R) = I_S(z,R) + I_{P,Res}(z,R)$, with the residual term $I_{P,Res}(z,R)$ satisfying

$$|I_{P,Res}(z,R)| \le C(1+|z|)^2 R^{-1/3} \quad as\ R \to +\infty.$$

Further, we have $I_S(z,R) = I_F(z) + I_{S,Res}(z,R)$ with the residual term $I_{S,Res}(z,R)$ satisfying

$$|I_{S,Res}(z,R)| \le C(1+|z|)^3 R^{-1/4} \quad as\ R \to +\infty.$$

- Remark: If R is large enough, then $I_P(z,R) \approx I_S(z,R)$.

Direct imaging method for recovering the interface

The properties of $I_p(z, R)$

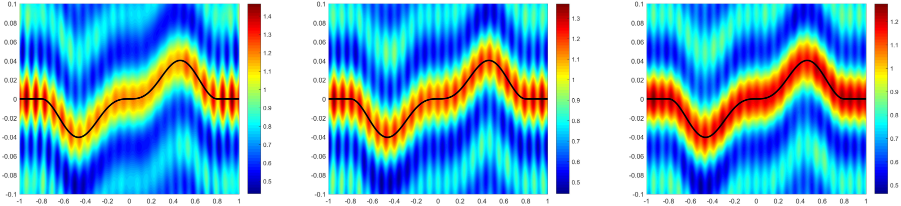
▶ It is expected that $I_S(z, R)$ takes a large value when $z \in \Gamma$ and decays as z moves away from Γ.

▶ If R is sufficiently large,

$$I_P(z, R) \approx I_S(z, R) = \int_{\partial B_R^+} |U(x, z)|^2 dx.$$

Thus, if R is sufficiently large, it is expected that $I_P(z, R)$ takes a large value when $z \in \Gamma$ and decays as z moves away from Γ.

Example 1:



(a) 10% noise, $k_+ = 40$, $k_- = 80$, $R = 1.5$

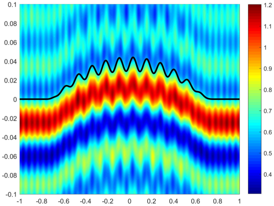
(b) 10% noise, $k_+ = 40$, $k_- = 80$, $R = 2$

(c) 10% noise, $k_+ = 40$, $k_- = 80$, $R = 3$

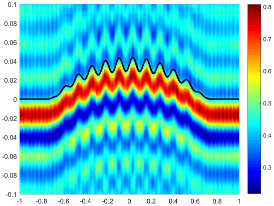
Figure: (a), (b) and (c) show the imaging results of $I_P(z, R)$ with the measured phaseless total-field data for different values of the radius R. The solid line represents the actual curve.

Numerical examples for recovering the interface

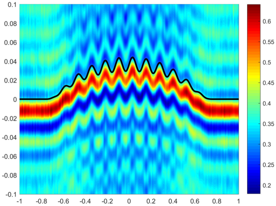
Example 2:



(a) 10% noise, $k_+ = 60$, $k_- = 30$, $R = 3$



(b) 10% noise, $k_+ = 90$, $k_- = 45$, $R = 3$



(c) 10% noise, $k_+ = 120$, $k_- = 60$, $R = 3$

Figure: Imaging results of $I_P(z, R)$ with the measured phaseless total-field data for different values of the wave numbers k_+ and k_-. The solid line represents the actual curve.

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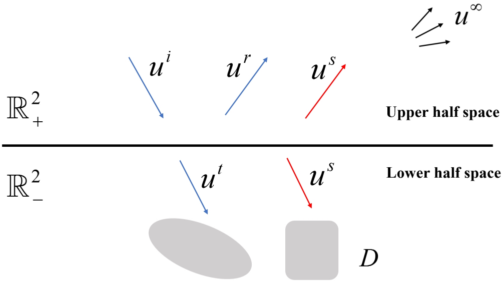
Direct scattering problem: given an incident plane wave u^i, find the total field $u^{tot} = u^0 + u^s$ and the scattered field u^s such that

$$\begin{aligned} \Delta u^{tot} + k_\pm^2 u^{tot} &= 0 \quad \text{in} \quad \mathbb{R}^2_\pm, \\ [u^{tot}] = 0, \quad \left[\partial u^{tot}/\partial \nu\right] &= 0 \quad \text{on} \quad \Gamma_0 := \{(x_1, 0) : x_1 \in \mathbb{R}\}, \\ u^{tot} &= 0 \quad \text{on} \quad \partial D, \\ \lim_{|x|\to+\infty} \sqrt{|x|}\left(\frac{\partial u^s}{\partial |x|} - ik_\pm u^s\right) &= 0 \quad \text{uniformly for all } \hat{x} = x/|x| \in \mathbb{S}^1_\pm, \end{aligned}$$



Inverse scattering by buried obstacles

- The inverse scattering problem with phased data

Determine the buried obstacles D by using the scattered field $u^s(x,d)$ or the far-field pattern $u^\infty(\widehat{x},d)$ with

$$u^s(x,d) = \frac{e^{ik_+|x|}}{\sqrt{|x|}} u^\infty(\widehat{x},d) + o\left(\frac{1}{\sqrt{|x|}}\right), \quad |x| \to \infty, \quad x \in \mathbb{R}^2_+,$$

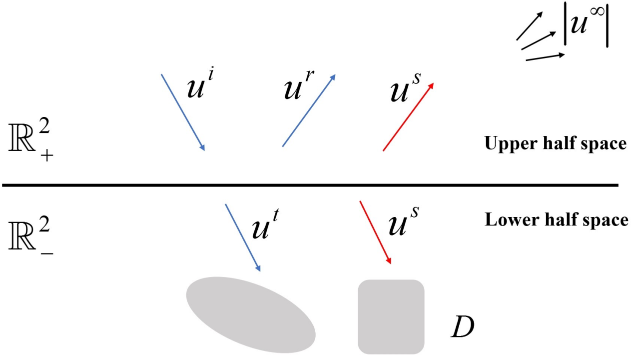
Here $u^\infty(\widehat{x},d)$ is called the far-field pattern of the scattered field $u^s(x,d)$.

- Existing numerical algorithms with phased data

Coyle' 00, Gebauer-Hanke-Kirsch-Muniz-Schneider' 05, Iakovleva-Ammari-Lesselier' 05, Delbary-Erhard-Kress-Potthast-Schulz' 08, Park' 10, Li-Li-Liu-Liu' 15, ...

Imaging of obstacles with phaseless far-field data

Reconstruct the buried obstacle D using the phaseless far-field data $|u^\infty|$.



Imaging of obstacles with phaseless far-field data

Translation invariance property of phaseless far-field data

Lemma 1.2 (J. Li, P. Li, H. Liu and X. Liu, Inverse Problems, 2015)

Define $D_z := D + z$ *with* $z = (z_1, 0)$, $z_1 \in \mathbb{R}$. *For the incident wave* $u^i(x, d)$ *with* $d \in \mathbb{S}^-_{\theta_c} := \{d = (\cos\theta_d, \sin\theta_d) : \theta_d \in [\pi + \theta_c, 2\pi - \theta_c]\}$, *the far-field patterns* $u^\infty(\cdot, d, D)$ *and* $u^\infty(\cdot, d, D_z)$ *associated with the obstacles* D *and* D_z, *respectively, satisfy*

$$u^\infty(\hat{x}, d, D_z) = e^{ik_-(z-\hat{x})\cdot d^t} u^\infty(\hat{x}, d, D), \quad \hat{x} \in \mathbb{S}^+_{\theta_c},$$

where $\hat{x} \in \mathbb{S}^+_{\theta_c} := \{\hat{x} = (\cos\theta_{\hat{x}}, \sin\theta_{\hat{x}}) : \theta_{\hat{x}} \in [\theta_c, \pi - \theta_c]\}$. *Here,* $\theta_c = 0$ *for the case of* $k_+ < k_-$ *and* $\theta_c = \arccos(k_-/k_+)$ *for the case of* $k_+ > k_-$.

Imaging of obstacles with phaseless far-field data

Break the translation invariance of phaseless far-field data [4] [5] [6] [7]

- Use the following superposition of two plane waves as the incident field:

$$u^i(x, d_1, d_2) := u^i(x, d_1) + u^i(x, d_2) = e^{ik_+ x\cdot d_1} + e^{ik_+ x\cdot d_2}$$

with the incident directions $d_1, d_2 \in \mathbb{S}^-_{\theta_c}$.

- The corresponding far field pattern is given by

$$u^\infty(x, d_1, d_2) := u^\infty(x, d_1) + u^\infty(x, d_2)$$

[4] B. Zhang and H. Zhang, *J. Comput. Phys.* 345 (2017), 58-73.

[5] B. Zhang and H. Zhang, *Inverse Problems* 34 (2018), 104005.

[6] X. Xu, B. Zhang and H. Zhang, *SIAM J. Appl. Math.* 78 (2018), 1737-1753.

[7] X. Xu, B. Zhang and H. Zhang, *SIAM J. Appl. Math.* 78 (2018), 3024-3039.

Direct imaging method for locating small scatterers

Indicator function:

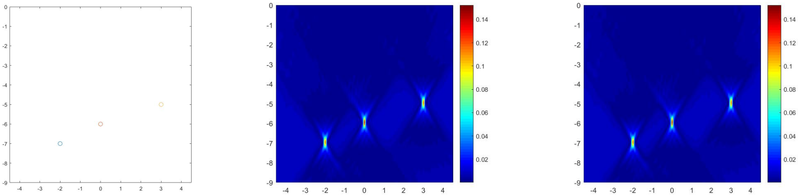
$$
\begin{aligned}
& I(z) \\
= & \int_{\mathbb{S}^{+}_{\theta_c}} \int_{\mathbb{S}^{-}_{\theta_c}} \int_{\mathbb{S}^{-}_{\theta_c}} |u^{\infty}(\hat{x}, d_1, d_2)|^2 \mathcal{T}(\theta_{d_1}) e^{-ik_- z\cdot d_1^t} \mathcal{T}(\theta_{d_2}) e^{ik_- z\cdot d_2^t} ds(d_1) ds(d_2) ds(\hat{x}) \\
& - \int_{\mathbb{S}^{-}_{\theta_c}} \mathcal{T}(\theta_d) e^{ik_- z\cdot d^t} ds(d) \int_{\mathbb{S}^{+}_{\theta_c}} \int_{\mathbb{S}^{-}_{\theta_c}} |u^{\infty}(\hat{x}, d)|^2 \mathcal{T}(\theta_d) e^{-ik_- z\cdot d^t} ds(d) ds(\hat{x}) \\
& - \int_{\mathbb{S}^{-}_{\theta_c}} \mathcal{T}(\theta_d) e^{-ik_- z\cdot d^t} ds(d) \int_{\mathbb{S}^{+}_{\theta_c}} \int_{\mathbb{S}^{-}_{\theta_c}} |u^{\infty}(\hat{x}, d)|^2 \mathcal{T}(\theta_d) e^{ik_- z\cdot d^t} ds(d) ds(\hat{x}),
\end{aligned}
$$

Properties of $I(z)$

$I(z)$ will take a large value in the neighborhood of $\partial D \cup \partial D'$ and decay as z moves away from $D \cup D'$, where $D' := \{-x : x \in D\}$

Direct imaging method for locating small scatterers

Example 3. Locating multiple small anomalies.

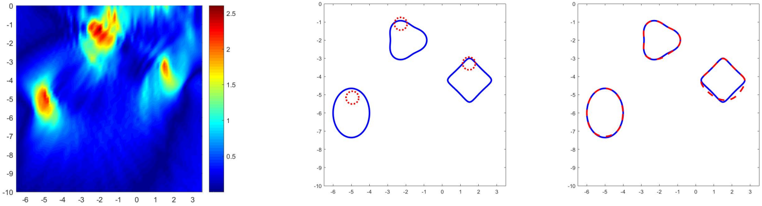


(a) (b) (c)

Figure: Imaging results of multiple small scatterers by direct imaging method with phaseless far-field data with (b) 5% noise and (c) 10% noise for the case $k_+ = 10\pi$ and $k_- = 1.45k_+$, where (a) shows the true scatterers.

Newton iteration method for extended obstacles

Example 4. Reconstruction of multiple extended obstacles



(a) (b) (c)

Figure: Location and shape reconstruction of multiple obstacles from the phaseless far-field data with 4% noise in the case $k_+ > k_-$. (a) The reconstruction result by the direct imaging method at $k_+^{(1)} = 30$ and $k_-^{(1)} = k_+^{(1)}/2$. (b) The initial curve for Newton iteration algorithm. (c) The reconstructed obstacle by Newton iteration algorithm at $k_+^{(2)} = 30$ and $k_-^{(2)} = k_+^{(2)}/2$.

Contents

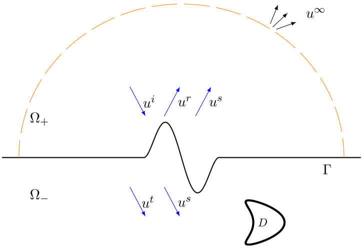
3. Scattering by locally rough surfaces and buried obstacles

Given an incident plane wave u^i, find the total-field $u^{tot} = u^0 + u^s$ and the scattered-field u^s such that

$$\begin{cases} \Delta u^{tot} + k_\pm^2 u^{tot} = 0 \quad \text{in } \Omega_\pm, \\ [u^{tot}] = 0, \quad \left[\partial u^{tot}/\partial\nu\right] = 0 \quad \text{on } \Gamma, \\ \mathscr{B}(u^{tot}) = 0 \quad \text{on } \partial D, \\ \lim\limits_{|x|\to+\infty} \sqrt{|x|}\left(\dfrac{\partial u^s}{\partial|x|} - ik_\pm u^s\right) = 0, \end{cases} \qquad \begin{cases} \Delta u^{tot} + k_+^2 u^{tot} = 0 \quad \text{in } \Omega_+, \\ \Delta u^{tot} + k_-^2 n(x) u^{tot} = 0 \quad \text{in } \Omega_-, \\ [u^{tot}] = 0, \quad \left[\partial u^{tot}/\partial\nu\right] = 0 \quad \text{on } \Gamma, \\ \lim\limits_{|x|\to+\infty} \sqrt{|x|}\left(\dfrac{\partial u^s}{\partial|x|} - ik_\pm u^s\right) = 0, \end{cases}$$

where $\mathscr{B}$ denotes the boundary conditions and $n \in L^\infty(D)$ is the refractive index with $\Re(n(x)) > 0$, $\Im(n(x)) \geq 0$ and $n \equiv 1$ in $\Omega_- \backslash \overline{D}$.

- both the buried obstacle and the locally rough surface exist



Inverse scattering by surfaces and obstacles

- Asymptotic behavior of u^s:

$$u^s(x,d) = \frac{e^{ik_+|x|}}{\sqrt{|x|}} u^\infty(\widehat{x},d) + o\left(\frac{1}{\sqrt{|x|}}\right), \quad |x| \to \infty, \quad x \in \Omega_+$$

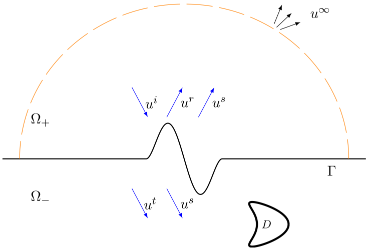
- $V_\pm$ are open subsets of $\mathbb{S}_1^\pm$.

Uniqueness Problem

Can $(\Gamma, D, \mathscr{B})$ and (Γ, n) be determined by $u^\infty(\hat{x}, d)$ with $(\hat{x}, d) \in V_+ \times V_-$?

- Uniqueness results for point source incidence

Liu-Zhang' 10, Li-Wang-Zhang' 19,
Li-Yang-Zhang' 22



Rellich lemma for the limited aperture case

- Let Γ_p denote the local perturbation of Γ.
- Let $R_0 > 0$ be large enough such that $\Gamma_p \cup \partial D \subset B_{R_0}$.
- We call $v \in H^1(\mathbb{R}^2 \setminus \overline{B_{R_0}})$ a radiating solution of Helmholtz equation in a two-layered medium if v satisfies

$$
\begin{aligned}
\Delta v + k_\pm^2 v = 0, &\qquad \text{in } \Omega_\pm^2 \backslash \overline{B_{R_0}}, &(1)\\
[v] = 0,\ [\partial_{x_2} v] = 0, &\qquad \text{on } \Gamma_0 \cap (\Omega^2 \setminus \overline{B_{R_0}}), &(2)\\
\lim_{|x| \to +\infty} \sqrt{|x|}\left(\partial_{|x|} v - ik_\pm v\right) = 0, &\qquad \text{uniformly for all } \hat{x} \in \mathbb{S}_\pm^1. &(3)
\end{aligned}
$$

Rellich lemma for the limited aperture case

- Let $\hat{x} := (\cos\theta_{\hat{x}}, \sin\theta_{\hat{x}})$ with $\theta_{\hat{x}} \in (0, \pi)$.
- The far-field pattern of the two-layered Green function:

$$G^\infty(\hat{x}, y) := \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k_+}} \begin{cases} e^{-ik_+\hat{x}\cdot y} + \mathcal{R}(\theta_{\hat{x}})e^{-ik_+\hat{x}\cdot y'}, & \hat{x} \in \mathbb{S}^1_+, \ y \in \mathbb{R}^2_+, \\ \mathcal{T}(\theta_{\hat{x}})e^{-ik_+(y_1\cos\theta_{\hat{x}} + iy_2\mathcal{S}(\cos\theta_{\hat{x}}, n))}, & \hat{x} \in \mathbb{S}^1_+, \ y \in \mathbb{R}^2_-. \end{cases}$$

- Far-field pattern of v:

$$v^\infty(\hat{x}) = \int_{\partial B_{R_0}} \left[\frac{\partial G^\infty(\hat{x}, y)}{\partial \nu(y)} v(y) - \frac{\partial v(y)}{\partial \nu(y)} G^\infty(\hat{x}, y) \right] ds(y), \quad \hat{x} \in \mathbb{S}^1_+.$$

Theorem 4 (L. Li, B. Zhang and H. Zhang' 23)

Let v be a radiating solution of the scattering problem (1)–(3). If $v^\infty = 0$ on some open subset of $\mathbb{S}^1_+$, then we have $v = 0$ in $\mathbb{R}^2 \setminus \overline{B_{R_0}}$.

Uniqueness theorem

Theorem 5 (L. Li, B. Zhang and H. Zhang' 23)

Let u_j^∞ $(j = 1, 2)$ *denote the far-field patterns for plane wave incidence corresponding to* Γ_j *and* D_j *with the boundary condition* $\mathscr{B}_j$.

If $u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d)$ *for* $(\hat{x}, d) \in V_+ \times V_-$, *then we have* $\Gamma_1 = \Gamma_2$, $D_1 = D_2$ *and* $\mathscr{B}_1 = \mathscr{B}_2$.

Uniqueness theorem

Theorem 6 (L. Li, B. Zhang and H. Zhang' 23)

Let u_j^∞ $(j = 1, 2)$ denote the far-field patterns for plane wave incidence corresponding to Γ_j and n_j.

If $u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d)$ for $(\hat{x}, d) \in V_+ \times V_-$, then we have $\Gamma_1 = \Gamma_2$ and $n_1 = n_2$.

Contents

Outline

1. Research on the inverse scattering problems in a two-layered medium
   - Imaging of the penetrable locally rough surface from phaseless total-field data
   - Imaging of buried obstacles from phaseless far-field data
   - Uniqueness for inverse scattering problems with limited aperture far-field measurements
2. Piecewise-analytic interfaces with weakly singular points of arbitrary order always scatter
3. Summary

4. Scattering in an inhomogeneous medium

- Given an incident field u^i, which is governed by the Helmholtz equation $(\Delta + k^2)u^i = 0$ in $\mathbb{R}^2$, find $u^{tot} = u^s + u^i$ such that the total field u^{tot} and the scattered field u^s satisfy

$$\begin{aligned} \Delta u^{tot} + k^2 q u^{tot} &= 0 \qquad \text{in} \quad \mathbb{R}^2, \\ \lim_{|x|\to+\infty} \sqrt{|x|}\left(\frac{\partial u^s}{\partial |x|} - iku^s\right) &= 0 \qquad \text{uniformly for all } \hat{x} = x/|x| \in \mathbb{S}^1, \end{aligned}$$

where $1 - q$ is compactly supported and $D = \{x \in \mathbb{R}^2 : 1 - q \neq 0\}$.

- It is known that u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} u^\infty(\widehat{x}, d) + O\left(|x|^{-3/2}\right), \quad |x| \to \infty, \quad x \in \mathbb{R}^2,$$

where $u^\infty(\widehat{x})$ is called the far-field pattern of the scattered field $u^s(x)$.

Non-scattering energy

If a penetrable obstacle D scatters some incoming wave trivially at the wavenumber $k > 0$, then k is called a non-scattering energy.

- Corners always scatter (curvilinear polygonal/polyhedral corner)
  - CGO solutions: Blåsten-Päivärinta-Sylvester' 14, Hu-Salo-Vesalainen' 16, Päivärinta-Salo-Vesalainen' 17, Blåsten' 18
  - Expansion methods: Elschner-Hu' 15, Elschner-Hu' 18
  - Free boundary methods: Cakoni-Vogelius' 21, Salo-Shahgholian' 21

Weakly singular points

Definition of weakly singular points

The point $O \in \partial D$ is called a weakly singular point of order $m \geq 2$ ($m \in \mathbb{N}$) if the subboundary $B_\epsilon(O) \cap \partial D$ for some $\epsilon > 0$, after a necessary coordinate translation and rotation, can be parameterized by the piecewise polynomial $x_2 = f(x_1), x_1 \in (-\epsilon/2, \epsilon/2)$, where

$$f(x_1) = \begin{cases} \sum_{l \in \mathbb{N}_0} \frac{f_l^+}{l!} x_1^l, & -\epsilon/2 < x_1 \leq 0, \\ \sum_{l \in \mathbb{N}_0} \frac{f_l^-}{l!} x_1^l, & 0 \leq x_1 < \epsilon/2. \end{cases} \quad (4)$$

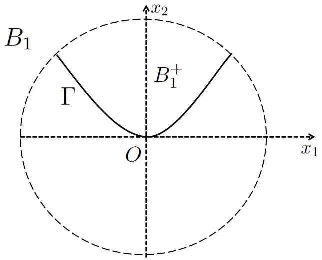
Here, the real-valued coefficients $\{f_l^\pm\}_{l=1}^\infty$ satisfy the relations

$$f_l^+ = f_l^- := f_l, \quad \forall\, 0 \leq l < m \quad \text{and} \quad f_m^+ \neq f_m^-,$$

with $f_l = 0$ for $l = 0, 1$. Moreover, the series (4) in $x_1 \geq 0$ (resp. $x_1 \leq 0$) converges at $x_1 = 0$.

Weakly singular points always scatter

- ∂D is at least C^1 smooth around the weakly singular points.



- Assumptions: q is analytic in $\overline{D}$ and $|q(O) - 1| + |\partial_1 q(O)| > 0$.

Theorem 7 (L. Li, G. Hu and J. Yang, JFA, 2023)

The penetrable scatterer $D \subset \mathbb{R}^2$ scatters every incoming wave, if ∂D contains at least one weakly singular point O. Further, u cannot be analytically continued from $\mathbb{R}^2 \backslash \overline{D}$ to $B_\epsilon(O)$ for any $\epsilon > 0$.

Local uniqueness results

Theorem 8 (L. Li, G. Hu and J. Yang, JFA, 2023)

Let D_j $(j = 1, 2)$ be two penetrable scatterers in $\mathbb{R}^2$ with the analytical potential functions q_j, respectively. If ∂D_2 differs from ∂D_1 in the presence of a weakly singular point lying on the boundary of the unbounded component of $\mathbb{R}^2 \backslash \overline{(D_1 \cup D_2)}$, then the far-field patterns corresponding to (D_j, q_j) incited by any non-vanishing incoming wave cannot coincide.

Contents

Conclusions

- A direct imaging method is developed to reconstruct the locally rough surfaces from phaseless total-field data.
- A direct imaging method is proposed to determine the location of small obstacles buried in the lower-half space from phaseless far-field data.
- The uniqueness is obtained for simultaneously determining locally rough surfaces and buried obstacles in a two-layered medium, with only limited-aperture far-field data measured.
- Piecewise-analytic interfaces with weakly singular points of arbitrary order always scatter.

Ongoing and future work

- To simultaneously reconstruct both the surface and buried obstacles.
- To prove the uniqueness for a locally rough surface with impedance condition by using limited-aperture far-field data (passive).
- How about the case of more than two layers?
- Do weakly singular points scatter for scattering by impenetrable obstacles?

Related works

Part I:

- Long Li, Jiansheng Yang, Bo Zhang and Haiwen Zhang, Imaging of buried obstacles in a two-layered medium with phaseless far-field data, *Inverse Problems* **37**(5) (2021) 055004 (26pp)
- Long Li, Jiansheng Yang, Bo Zhang and Haiwen Zhang, Uniform far-field asymptotics of the two-layered Green function in 2D and application to wave scattering in a two-layered medium, arXiv:2208.00456.
- Long Li, Jiansheng Yang, Bo Zhang and Haiwen Zhang, A direct imaging method is developed to reconstruct the locally rough surfaces from phaseless total-field data, in preparation.
- Long Li, Bo Zhang and Haiwen Zhang, Uniqueness for inverse scattering problems with limited aperture far-field measurements in a two-layered medium, in preparation.

Part II:

- Long Li, Guanghui Hu and Jiansheng Yang, Piecewise-analytic interfaces with weakly singular points of arbitrary order always scatter, Journal of Functional Analysis, 284 (2023), 109800 (31pp)

Thank you!