Log-stability results for inverse coefficients problem associated with time harmonic magnetic Schrödinger operator

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Plan

1. Introduction

Statement of the problem

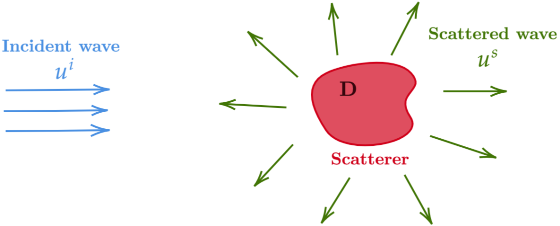
- Let $D \subset \mathbb{R}^3$, be a bounded open set with smooth boundary such that $(\mathbb{R}^3 \setminus D)$ is connected.
- Let $B = B(0, a) \supset D$, $a > 0$.
- Let the magnetic potential $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ such that $\mathrm{supp}(A) \subset D$.
- Let the electric potential $q \in L^\infty(\mathbb{R}^3, \mathbb{C})$ such that $\mathrm{Im}(q) \geq 0$, $\mathrm{supp}(q) \subset D$.
- We deal with a magnetic Schrödinger operator in three-dimentional case

$$\mathcal{H}_{A,q} = -(\nabla + iA(x))^2 + q(x) \equiv -\Delta - Q_{A,q}, \quad x \in \mathbb{R}^3, \tag{1}$$

where $Q_{A,q}$ is a first order operator given by

$$Q_{A,q}v(x) = i\mathrm{div}(A(x)v(x)) + iA(x) \cdot \nabla v(x) - (|A(x)|^2 + q(x))v(x), \quad v \in H^1_{loc}(\mathbb{R}^3). \tag{2}$$

Statement of the problem



We introduce the following scattering problem: Given an incident field $u^i \in H^1(D)$, find a total field u such that

$$\begin{cases} \mathcal{H}_{A,q}u(x) - k^2 u(x) = 0, & x \in \mathbb{R}^3, \\ u(x) = u^i(x) + u^s(x), & x \in \mathbb{R}^3, \\ \lim_{r\to\infty} r\left(\partial_r u^s - iku^s\right) = 0, & r = |x|, \end{cases} \tag{3}$$

where $u^s \in H^2_{loc}(\mathbb{R}^3)$ is the scattered field and k is the wave number.

$\Rightarrow$ We will treat **two inverse problems** for the stable determination of the magnetic potential and the electric potential appearing in (3) from scattered field measurements.

Well-posedness

- The total field u satisfies the so-colled **Lippmann-Schwinger equation**

$$u(x) = u^i(x) + \int_{\mathbb{R}^3} \Phi(x,y) Q_{A,q} u(y)\, dy, \quad x \in \mathbb{R}^3, \tag{4}$$

where Φ denotes the fundamental solution to the Helmholtz equation

$$\Phi(x,y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y. \tag{5}$$

and

$$Q_{A,q} u(x) = i\mathrm{div}(A(x)u(x)) + iA(x) \cdot \nabla u(x) - (|A(x)|^2 + q(x))u(x).$$

K. Krupchyk and G. Uhlmann, *Uniqueness in an inverse boundary problem for a magnetic Schrödinger operator with a bounded magnetic potential*, Comm. Math. Phys., 327, pp. 993-1009, (2014).

V. Serov, and J. Sandhu, *Scattering solutions and Born approximation for the magnetic Schrödinger operator*, Inverse Problems in Science and Engineering, 27, Issue 4, 422 - 438, (2019).

$\Rightarrow$ The Lippmann-Schwinger equation has a unique scattering solution u such that $u^s \in H^1_{loc}(\mathbb{R}^3)$.

The problem is well-posed

Known results

- J. Sylvester, G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. 125 (1987), 153-169.

  ⟶ uniqueness result for the D-to-N map based on geometric optical solutions implies uniqueness at a fixed energy for compactly supported potentials.

- P. Hähner, and T. Hohage, *New stability estimates for the inverse acoustic inhomogeneous medium problem and applications*, SIAM journal on mathematical analysis, 33(3), 670-685, (2001).

  ⟶ Logarithmic stability for q when $A = 0$ from the far field pattern.

- L. Tzou, *Stability estimates for coefficients of magnetic Schrödinger equation from full and partial boundary measurements*, Communication in Partial Differential Equations 33, 1911-1952, (2008).

  ⟶ Stability result for magnetic schrodinger equation from the corresponding global Dirichlet to Neumann map.

- H. Ben Joud, *A stability estimate for an inverse problem for the Schrödinger equation in a magnetic field from partial boundary measurements*, Inverse Problems 25, 045012 (23 pp), (2009).

  ⟶ Determining A and q of the magnetic schrödinger equation from D-to-N map.

2. Stability analysis for near field data

The direct problem in the near field setting

Let $y \in \partial B$ be the location of a point source.

The total field $u(\cdot, y)$ generated by the point source satisfies

$$\mathcal{H}_{A,q} u(\cdot, y) - k^2 u(\cdot, y) = \delta_y \quad \text{in } \mathbb{R}^3, \tag{6}$$

$$u(\cdot, y) = \Phi(\cdot, y) + u^s_{A,q}(\cdot, y) \quad \text{in } \mathbb{R}^3, \tag{7}$$

$$\lim_{r \to \infty} r \left(\partial_r u^s_{A,q} - ik u^s_{A,q} \right) = 0, \; r = |x|, \tag{8}$$

where

- the scattered field $u^s_{A,q}(\cdot, y) \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$.
- the incident field is given by

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y, \tag{9}$$

  and is the fundamental solution of the Helmholtz equation.
- δ_y is the Dirac distribution at y.

$\Rightarrow$ **The problem is well-posed**

The near field operator

We define **the near field operator** $\mathcal{N}_{A,q} : L^2(\partial B) \to L^2(\partial B)$, as

$$\mathcal{N}_{A,q} h(x) := \int_{\partial B} u^s_{A,q}(x, y) h(y) \, \mathrm{ds}(y), \quad x \in \partial B, \tag{10}$$

where $u^s_{A,q}(\cdot, y)$ is given by (7) and satisfying the Sommerfeld radiation condition (8).

$\Rightarrow$ The **inverse problem** that we shall consider in the near field setting is to recover A and q from the the near field operator $\mathcal{N}_{A,q}$.

Gauge invariance

$\Rightarrow$ The magnetic potential A cannot be uniquely determined from near field measurements outside B.

- Given $\varphi \in W^{2,\infty}(\mathbb{R}^3)$, $\mathrm{supp}(\varphi) \subset B$ and let $\tilde{u} = ue^{-i\varphi}$.

$$\Rightarrow \mathcal{H}_{A+\nabla\varphi,q}\tilde{u} = e^{-i\varphi(x)}\mathcal{H}_{A,q}u. \tag{11}$$

- From the uniqueness of solutions and the fact that $\varphi = 0$ outside B, we deduce that $\forall y \in \partial B$

$$u^s_{A+\nabla\varphi,q}(\cdot,y) = (e^{-i\varphi(x)} - 1)\Phi(\cdot,y) + e^{-i\varphi(x)}u^s_{A,q}(\cdot,y) \quad \text{in } \mathbb{R}^3,$$

$$\Rightarrow u^s_{A+\nabla\varphi,q}(\cdot,y) = u^s_{A,q}(\cdot,y) \quad \text{outside } D.$$

$\Rightarrow$ Thanks to the identity $\mathrm{curl}(A) = \mathrm{curl}(A+\nabla)$, where

$$\mathrm{curl}(A) = \sum_{j,\ell=1}^{3}\left(\frac{\partial a_j}{\partial x_\ell} - \frac{\partial a_k}{\partial x_j}\right)dx_j \wedge dx_\ell, \quad A = (a_j)_{1\leq j\leq 3}.$$

$\Rightarrow$ **Goal:** To determine $\mathrm{curl}(A)$ and q from the knowledge of the near field operator $\mathcal{N}_{A,q}$.

Definitions and notations

- Let $M > 0$, $\sigma > 0$ and $\gamma > 0$ be given.
- Let define

* The class of admissible magnetic potentials $\mathcal{A}_\sigma(M)$ by

$$\mathcal{A}_\sigma(M) := \Big\{ A \in W^{2,\infty}(\mathbb{R}^3, \mathbb{R}^3),\ \mathrm{Supp}(A) \subset D,\ \|A\|_{W^{2,\infty}} \leq M, \text{ and } \|\widehat{\mathrm{curl} A}\|_{L^1_\sigma(\mathbb{R}^3)} \leq M \Big\}, \quad (12)$$

where $L^1_\tau(\mathbb{R}^3)$, $\tau > 0$, be the weighted $L^1(\mathbb{R}^3)$ space with norm

$$\|v\|_{L^1_\tau(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^{\tau/2} |v(\xi)| d\xi.$$

* The class of admissible electric potentials $\mathcal{Q}_\gamma(M)$ by

$$\mathcal{Q}_\gamma(M) := \Big\{ q \in L^\infty(\mathbb{R}^3, \mathbb{C}),\ \mathrm{Im}(q) \geq 0,\ \mathrm{Supp}(q) \subset D,\ \|q\|_{L^\infty(D)} \leq M \text{ and } \|\widehat{q}\|_{L^1_\gamma(\mathbb{R}^3)} \leq M \Big\}. \quad (13)$$

Main results

Theorem (Stability estimates)

Let $M > 0$, $\sigma > 0$ and $\gamma > 0$. Then there exists a constant $C > 0$ such that for any $(A_j, q_j) \in \mathcal{A}_\sigma(M) \times \mathcal{Q}_\gamma(M)$, $j = 1, 2$, we have

$$\|\mathrm{curl}(A_1) - \mathrm{curl}(A_2)\|_{L^\infty(D)} \leq C\left(\kappa^{1/2} + |\log(\kappa)|^{-\frac{\sigma}{(\sigma+3)}}\right), \tag{14}$$

and

$$\|q_2 - q_1\|_{L^\infty(D)} \leq C\left(\kappa^{1/2} + |\log(\kappa)|^{-\frac{\gamma\sigma}{(\sigma+3)(2\gamma+3)}}\right), \tag{15}$$

where $\kappa = \|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\|$. Here C depends only on B, M, σ and γ.

Corollary 1 (Uniqueness)

Let $A_1, A_2 \in \mathcal{A}_\sigma(M)$, $q_1, q_2 \in \mathcal{Q}_\gamma(M)$ and $B \supset D$. Then, we have

$$u^s_{A_1,q_1}(x, y) = u^s_{A_2,q_2}(x, y), \quad \forall (x, y) \in \partial B \times \partial B,$$

implies $q_1 = q_2$ and $\mathrm{curl}\, A_1 = \mathrm{curl}\, A_2$ in D.

Construction of the solution: Method of geometrical optics

- Let $\omega = \omega_1 + i\omega_2$ be a vector with $\omega_1, \omega_2 \in \mathbb{S}^2$, and $\omega_1 \cdot \omega_2 = 0$.
- Let $N_\omega = \omega \cdot \nabla$.

Lemma 1

Let $A \in W^{2,\infty}(D)$ and $q \in L^\infty(D)$ such that $\|A\|_{W^{2,\infty}(D)} \leq M$, $\|q\|_{L^\infty(D)} \leq M$ for $M > 0$, and $\mathrm{Supp}(A)$, $\mathrm{Supp}(q) \subset D$. There exists $s_0 > 0$ such that for any $s \geq s_0$, $\rho = s\omega \in \mathbb{C}$ satisfying $\rho \cdot \rho = 0$, there exist complex geometrical solution $u(\cdot, \rho) \in H^2(B)$ such that

$$u(x,\rho) = e^{ix\cdot\rho}(e^{i\varphi(x,\omega)} + r(x,\rho)), \tag{16}$$

to the equation $\mathcal{H}_{A,q} u = k^2 u$ in D, where $\varphi(x,\omega) = N_\omega^{-1}(-\omega \cdot A)$ and

$$\|r(\cdot,\rho)\|_{H^m(B)} \leq Cs^{m-1}, \quad 0 \leq m \leq 2 \quad \text{and} \quad \|u(\cdot,\rho)\|_{H^2(B)} \leq Cs^2 e^{\Lambda s}, \tag{17}$$

where C, Λ and s_0 depend only on B, k and M.

L. Tzou, *Stability estimates for coefficients of magnetic Schrödinger equation from full and partial boundary measurements*, Communication in Partial Differential Equations 33, 1911-1952, (2008).

Sketch of proof

Stability estimate for the magnetic field:

- Let $\xi \in \mathbb{R}^3$, ω_1, $\omega_2 \in \mathbb{S}^2$ be three mutually orthogonal vectors in $\mathbb{R}^3$.
- For each $s > \frac{|\xi|}{2}$, let

$$\rho_1 = s\left(i\omega_2 + \left(-\frac{\xi}{2s} + \sqrt{1-\frac{|\xi|^2}{4s^2}}\omega_1\right)\right) = s\omega_1^*(s), \tag{18}$$

$$\rho_2 = s\left(-i\omega_2 + \left(\frac{\xi}{2s} + \sqrt{1-\frac{|\xi|^2}{4s^2}}\omega_1\right)\right) = s\omega_2^*(s). \tag{19}$$

- For $s \geq s_0$ for some s_0 sufficiently large: u_1 solves $\mathcal{H}_{-A_1,q_1}u_1 = k^2u_1$ in B and u_2 solves $\mathcal{H}_{A_2,q_2}u_2 = k^2u_2$ in B and such that

$$u_j(x,\rho_j) = e^{ix\cdot\rho_j}(e^{i\varphi_j(x,\omega_j^*)} + r_j(x,\rho_j)), \tag{20}$$

where $r_j(\cdot,\rho_j)$, $j=1,2$ satisfies

$$\|r_j(\cdot,\rho_j)\|_{H^m(D)} \leq Cs^{m-1}, \quad 0 \leq m \leq 2, \tag{21}$$

and $\varphi_1(x,\omega_1^*) = N_{\omega_1^*}^{-1}(\omega_1^*\cdot A_1)$ and $\varphi_2(x,\omega_2^*) = N_{\omega_2^*}^{-1}(-\omega_2^*\cdot A_2)$ are solutions of

$$\omega_1^*\cdot\nabla\varphi_1(\cdot,\omega_1^*) = \omega_1^*\cdot A_1, \quad \omega_2^*\cdot\nabla\varphi_2(\cdot,\omega_2^*) = -\omega_2^*\cdot A_2. \tag{22}$$

Sketch of the proof

- Let $A(x) := (A_2 - A_1)(x), \quad q(x) := (q_2 - q_1)(x), \quad x \in \mathbb{R}^3$.
- Then for any $|\xi| \leq s$, we have the following identity

$$i \int_D A(x) \cdot (u_2 \nabla u_1 - u_1 \nabla u_2)\, dx = 2s \int_D \overline{\omega} \cdot A(x) e^{ix\cdot\xi} dx + \mathcal{R}(\xi, s), \qquad (23)$$

  with $|\mathcal{R}(\xi, s)| \leq C \langle \xi \rangle$.
- Let $a_j(x) = A(x) \cdot e_j, \quad j = 1, 2, 3, \quad x \in \mathbb{R}^3$ where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$.
- We set for $j, \ell = 1, 2, 3$,

$$b_{j\ell}(x) := \frac{\partial a_\ell}{\partial x_j}(x) - \frac{\partial a_j}{\partial x_\ell}(x), \quad x \in \mathbb{R}^3,$$

$$\hat{b}_{j\ell}(\xi) := \int_{\mathbb{R}^3} e^{ix\cdot\xi} b_{j\ell}(x) dx.$$

Sketch of the proof

Lemma 2: (Estimate of the Fourier transform)

For any $s \geq s_0$ and $\xi \in \mathbb{R}^3$ such that $|\xi| \leq s$ the following estimate holds true,

$$|\hat{b}_{j\ell}(\xi)| \leq C\langle\xi\rangle \left(e^{\Lambda s}\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\| + s^{-1}\langle\xi\rangle\right) \tag{24}$$

for $j, \ell = 1, 2, 3$, where C and Λ are positive constants independent of s, ξ and M.

Let $s_0 > 1$ and s and R be two parameters satisfying $s \geq R \geq s_0$.

- Using (24) and the fact that for $j = 1, 2$, $\int_{\mathbb{R}^3} \langle\xi\rangle^{\sigma} |\widehat{\mathrm{curl} A_j}(\xi)|\, d\xi < M$, for some $\sigma > 0$

$$\begin{aligned}\int_{\mathbb{R}^3} |\hat{b}_{j\ell}(\xi)|\, d\xi &= \int_{\langle\xi\rangle\leq R} |\hat{b}_{j\ell}(\xi)|\, d\xi + \int_{\langle\xi\rangle\geq R} |\hat{b}_{j\ell}(\xi)|\, d\xi \\ &\leq CR^2\left(e^{\Lambda s}\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\| + s^{-1}R\right) + 2MR^{-\sigma}.\end{aligned}$$

Sketch of the proof

- Choosing $R = s^{1/(\sigma+3)}$, we deduce that for s_0 sufficiently large

$$\|b_{j\ell}\|_{L^\infty(\mathbb{R}^3)} \leq C'\big(e^{\Lambda' s}\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\| + s^{-\sigma/\sigma+3}\big), \quad \forall s \geq s_0. \tag{25}$$

- If $\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\| \leq \varepsilon_0$, for some $\varepsilon_0 > 0$, such that $-\log(\varepsilon_0) \geq 2\Lambda' s_0$, then taking $s = \frac{-1}{2\Lambda'}\log(\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\|)$ in (25) implies

$$\|b_{j\ell}\|_{L^\infty(\mathbb{R}^3)} \leq C'\big(\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\|^{1/2} + \big(\frac{-1}{2\Lambda'}\log(\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\|)\big)^{-\sigma/\sigma+3}\big). \tag{26}$$

- This inequality holds true if $\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\| \geq \varepsilon_0$ and we can write

$$\|b_{j\ell}\|_{L^\infty(\mathbb{R}^3)} \leq M \leq (M/\sqrt{\epsilon_0})\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\|^{1/2}. \tag{27}$$

Sketch of proof

Stability estimate for the electric potential:

- Apply the Hodge decomposition to $A = A_1 - A_2$ in the space $W^{2,\infty}(D, \mathbb{R}^3)$. Then there exist $\varphi \in W^{2,\infty}(B)$ with $\mathrm{supp}(\varphi) \subset D$ such that

$$A = A_1 - A_2 = \tilde{A} + \nabla\varphi. \tag{28}$$

- We define

$$\tilde{A}_1 = A_1 - \frac{1}{2}\nabla\varphi, \quad \tilde{A}_2 = A_2 + \frac{1}{2}\nabla\varphi. \tag{29}$$

- Using Morrey's inequality, we get that $\tilde{A} = \tilde{A}_1 - \tilde{A}_2$ verifies

$$\|\tilde{A}\|_{W^{2,\infty}(B)} \leq C\|\mathrm{curl}A_1 - \mathrm{curl}A_2\|_{L^\infty(D)}, \tag{30}$$

- Due the gauge invariance of the scattered field and since $\varphi_{|\partial B} = 0$, we get

$$\mathcal{N}_{\tilde{A}_j, q_j} = \mathcal{N}_{A_j, q_j}, \quad j = 1, 2. \tag{31}$$

Sketch of the proof

- Using the method of geometrical optics, we construct u_j, $j = 1, 2$ (given by (20)) for some s_0. Using the Hodge decomposition and the Gauge invariance (31), we obtain

Lemma 3

There exists $s_0 > 0$ such that $\forall s \geq s_0$ and $\xi \in \mathbb{R}^3$ with $|\xi| \leq s$ the following estimate holds true,

$$|\widehat{q}(\xi)| \leq C\big(e^{\Lambda s}\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\| + s\|\mathrm{curl}(A)\|_{L^\infty(D)} + s^{-1}\langle\xi\rangle\big). \tag{32}$$

The constants s_0, C and Λ depend only on B, M and k.

- We assume that for $j = 1, 2$, $\int_{\mathbb{R}^3} \langle\xi\rangle^\gamma |\hat{q}_j(\xi)|\, d\xi < M$, for some $\gamma > 0$.

$\Rightarrow$ From the stability estimate for the magnetic field and for s_0 sufficiently large, we obtain the stability estimate for the electric potential.

3. Stability analysis for far field data

The direct scattering problem in the far field setting

The direct scattering problem in the far field setting formally corresponds with letting $|y| \to \infty$ in the direction $-d$ with $d \in \mathbb{S}^2$ and can be phrased as follows:

Given an incident plane wave $u^i(x,d) = e^{ikx\cdot d}$, $x \in \mathbb{R}^3$ where k is the wave number and $d \in \mathbb{S}^2$, seek a total field $u_{A,q}(\cdot,d)$ that satisfies

$$\left\{ \begin{array}{cc} \mathcal{H}_{A,q} u(\cdot,d) - k^2 u(\cdot,d) & \text{in } \mathbb{R}^3, \\ u(\cdot,d) = u^i(x,d) + u^s_{A,q}(\cdot,d) & \text{in } \mathbb{R}^3, \end{array} \right. \tag{33}$$

where the scattered field $u^s_{A,q}(\cdot,d) \in H^2_{\mathrm{loc}(\mathbb{R}^3)}$ and satisfies the Sommerfeld radiation condition.

The far field pattern

Representing $u^s_{A,q}(\cdot, d)$, $d \in \mathbb{S}^2$ in terms of the outgoing fundamental solution of $\Delta + k^2$, it follows that as $|x| \to \infty$

$$u^s_{A,q}(x,d) = \frac{e^{ik|x|}}{|x|}\left(u^\infty_{A,q}(\hat{x},d) + O\left(\frac{1}{|x|}\right)\right), \qquad \hat{x} = \frac{x}{|x|}, \tag{34}$$

where $u^\infty_{A,q}(\hat{x},d)$ is defined to be the **scattering amplitude** (or **far field pattern**).

$\Rightarrow$ The **inverse problem** that we shall consider in the far field setting is to recover A and q from $u^\infty_{A,q}(\hat{x},d)$, for all $\hat{x}, d \in \mathbb{S}^2$.

Gauge invariance

- Given $\varphi \in W^{2,\infty}(\mathbb{R}^3)$, $\text{supp}(\varphi) \subset B$ and let $\tilde{u} = u(x)e^{-i\varphi(x)}$

$$\mathcal{H}_{A+\nabla\varphi,q}\tilde{u} = e^{-i\varphi(x)}\mathcal{H}_{A,q}u. \tag{35}$$

- Since $\varphi = 0$ outside B and the uniqueness of solutions, we can deduce that for all $d \in \mathbb{S}^2$

$$u^s_{A+\nabla\varphi,q}(\cdot,d) = (e^{-i\varphi(x)} - 1)ui(\cdot,d) + e^{-i\varphi(x)}u^s_{A,q}(\cdot,d) \quad \text{in } \mathbb{R}^3,$$

- This shows that

$$u^s_{A+\nabla\varphi,q}(\cdot,d) = u^s_{A,q}(\cdot,d) \quad \text{outside } D.$$

$\Rightarrow$ The magnetic potential A cannot be uniquely determined from far field measurements outside B.

$\Rightarrow$ **Goal:** To determine $curl(A)$ and q from the far field $u^\infty_{A,q}(\hat{x},d)$, for all $(\hat{x},d) \in \mathbb{S}^2 \times \mathbb{S}^2$.

Main results

Theorem 2 (Stability estimates)

Let $M > 0$, $\sigma > 0$, $\gamma > 0$ and $\epsilon > 0$. Then there exist two constants $C > 0$ and $\delta > 0$ such that for all $(A_j, q_j) \in \mathcal{A}_\sigma(M) \times \mathcal{Q}_\gamma(M)$, $j = 1, 2$ verifying $\|u^\infty_{A_1,q_1} - u^\infty_{A_2,q_2}\|_{L^2(\partial B\times\partial B)} < \delta$, we have

$$\|\mathrm{curl}(A_1) - \mathrm{curl}(A_2)\|_{L^\infty(D)} \leq C|\log(\kappa)|^{-\frac{\sigma}{\sigma+3}+\epsilon}, \tag{36}$$

and

$$\|q_2 - q_1\|_{L^\infty(D)} \leq C|\log(\kappa)|^{-\frac{\gamma\sigma}{(\sigma+3)(2\gamma+3)}+\epsilon}, \tag{37}$$

where $\kappa = \|u^\infty_{A_1,q_1} - u^\infty_{A_2,q_2}\|_{L^2(\partial B\times\partial B)}$. Here C depends only on D, M, a, ϵ, σ, δ and γ.

Corollary 2 (Uniqueness)

Let A_1, $A_2 \in \mathcal{A}_\sigma(M)$, $q_1, q_2 \in \mathcal{Q}_\gamma(M)$. Then, we have

$$u^\infty_{A_1,q_1}(\hat{x}, d) = u^\infty_{A_2,q_2}(\hat{x}, d), \quad \forall(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2,$$

implies $q_1 = q_2$ and $\mathrm{curl}\, A_1 = \mathrm{curl}\, A_2$ in D.

Sketch of proof

Relation between the scattered field and the far field pattern:

Lemma 4

Let $A \in W^{1,\infty}(D, \mathbb{R}^3)$ and $q \in L^\infty(D, \mathbb{C})$ with $\mathrm{Supp}(A)$, $\mathrm{Supp}(q) \subset D$ and $\mathrm{Im}(q) \geq 0$. For $k > 0$ fixed, we have

$$u^s_{A,q}(x,y) = \frac{1}{4\pi}\frac{e^{ik|x|}}{|x|}\frac{e^{ik|y|}}{|y|}u^\infty_{A,q}(\widehat{x}, -\widehat{y}) + \frac{1}{|x||y|}\left(\frac{1}{|x|}+\frac{1}{|y|}\right)\Lambda(x,y), \qquad x \neq y, \tag{38}$$

where $\Lambda(x,y)$ is uniformly bounded as $|x| \longrightarrow \infty$ and $|y| \to \infty$.

Sketch of proof

Lemma 5

Let $M > 0$ and $0 < \theta < 1$ be given. Let $A_j \in W^{1,\infty}(D, \mathbb{R}^3)$ and $q_j \in L^\infty(D, \mathbb{C})$ such that $\|A_j\|_{W^{1,\infty}} \leq M$ and $\|q_j\|_{L^\infty} \leq M$. Then there exist a constants $\eta > 0$ that only depends on M, k, a and θ and a constant ω that only depends on a and k such that

$$\|\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}\| \leq \eta^2 \exp\Big(- \big(- \ln \frac{\|u^\infty_{A_1,q_1} - u^\infty_{A_2,q_2}\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}}{\omega\eta} \big)^\theta \Big),$$

where $\mathcal{N}_{A_j,q_j}$, $j = 1, 2$ denote here the near field operators associated with $B = \{x \in \mathbb{R}^3, |x| < 2a\}$.

P. Hähner, and T. Hohage, *New stability estimates for the inverse acoustic inhomogeneous medium problem and applications*. SIAM journal on mathematical analysis, 33(3), 670-685, (2001).

→ For the proof of the stability estimates, it's based by using the Theorem 1 and the Lemma 5.

Conclusions and perspectives

Conclusions:

- Stability estimates for the magnetic field and the electric potential from the far field pattern $u^{\infty}_{A,q}(\hat{x}, d)$, $\forall \hat{x}, d \in \mathbb{S}^2$.
- Stability estimates for the magnetic field and the electric potential from the near field operator $\mathcal{N}_{A,q}$.

Perspectives:

- The uniqueness of the reconstruction of the domain D for $q \neq 0$ and $A \neq 0$.
- Development analysis of sampling methods for the reconstruction of the D support from the knowledge of the far-field data.
- Analysis of the interior transmission problem for $A \neq 0$.

Thank you for your attention!