Fading regularization inverse methods for the identification of boundary conditions in thin plate theory

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Motivations : Identifications of boundary conditions in structural mechanics

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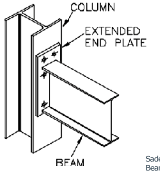
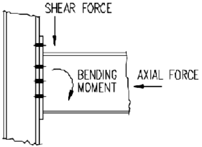
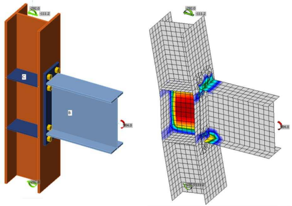
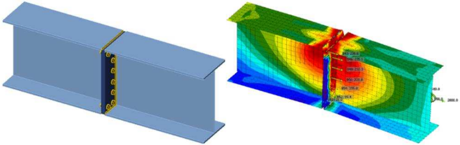




Sadeghian, Mojtaba et al. "Numerical Analysis of End Plate Bolted Connection with Corrugated Beam." *World Academy of Science, Engineering and Technology, International Journal of Civil, Environmental, Structural, Construction and Architectural Engineering* 9 (2015): 1496-1500.



https://www.ideastatica.com/support-center/extended-end-plate-moment-connections-aisc



https://www.ideastatica.com/support-center/bolted-plate-to-plate-en

https://www.energierecrute.com

1. The biharmonic Cauchy problem
   - Equivalent formulation of the problem
   - The fading regularization method
   - Convergence of the continuous formulation
   - Convergence of the discrete formulation
   - Stopping criteria
   - Numerical implementation using the MFS
   - Numerical implementations using the FEM
2. Cauchy problem in thin plate theory
3. Plate finite element for second order Cauchy problem
4. Conclusion and Outlooks

Cauchy problem associated with the biharmonic equation

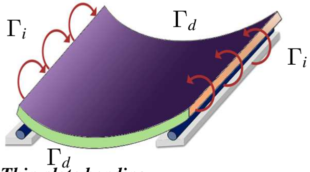
$$\Delta^2 u = 0 \quad \forall x \in \Omega$$

ou

$$\begin{cases} \Delta u = v & \forall x \in \Omega \\ \Delta v = 0 & \forall x \in \Omega \end{cases}$$

$\partial\Omega = \Gamma_d \cup \Gamma_i$ et $\Gamma_d \cap \Gamma_i = \emptyset$

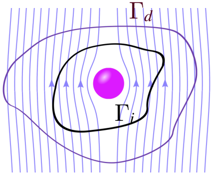
où $u_{,n} = \frac{\partial u}{\partial n}$ et $v_{,n} = \frac{\partial v}{\partial n}$



Thin plate bending

u : the deflection of the plate

v : the bending moment



Stokes flow

u : the stream function

v : the vorticity of the fluid

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with

$$\begin{cases} u = \varphi_d & \forall x \in \Gamma_d \\ u_{,n} = \psi_d & \forall x \in \Gamma_d \\ v = \mu_d & \forall x \in \Gamma_d \\ v_{,n} = \phi_d & \forall x \in \Gamma_d \end{cases}$$

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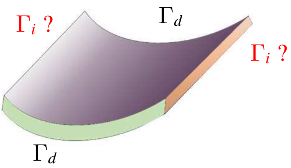
No boundary condition is given on Γ_i



→ ill-posed problem in the sens of Hadamard

the stability of the solution cannot be guaranteed

→ It's an inverse problem !

→ Cannot be solved by the usual methods

Examples of regularization methods

Based on a reformulation of the Cauchy problem :

- The method based on minimization of an energy-like error Functional (*Andrieux et al.* (2005-2006))

  Transform the problem into two well-posed problem with mixed boundary conditions and minimize the gap between the two field solutions.

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  Second order ill-posed Cauchy problem $\rightsquigarrow$ Fourth order well-posed problem

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- Fading regularization method (*Cimetière et al. (2000,2001), Delvare (2000)*)

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Equivalent formulation of the problem

For $\Phi_d = (\varphi_d, \psi_d, \mu_d, \phi_d)$ a quadruplet of compatible data on Γ_d, (i.e. $\Phi_d \in H(\Gamma_d)$), the biharmonic Cauchy problem is equivalent to :

$$\begin{cases} \mathbf{U} = (u, u_{,n}, v, v_{,n}) \in H(\Gamma) \text{ such as :} \\ \mathbf{U} = \Phi_d \qquad \text{on } \Gamma_d \end{cases}$$

with

$$H(\Gamma) = \big\{\Phi = (\varphi, \psi, \mu, \phi) \in X(\Gamma) \text{ such as } \exists u \in \mathscr{H}_0^2 \\ \text{with } v = \Delta u \text{ and } (u, u', v, v') = (\varphi, \psi, \mu, \phi)\big\}\,,$$

such as

$$X(\Gamma) = H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{-3/2}(\Gamma)$$

and

$$\mathscr{H}_0^2 = \{u \in H^2(\Omega) \quad / \quad \Delta^2 u = 0\}.$$

The fading regularization method

Cimetière et al. (2000,2001), Delvare (2000)

Basic idea : Seeking among all solutions of the equilibrium equation in Ω, the one that fits the best the boundary conditions available on Γ_d, with :

- independence to a regularization parameter,
- stability towards noisy data,

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1^{st} *intuition*

$$\text{Minimize } \|\mathbf{V} - \Phi_d\|^2_{\Gamma_d}, \quad \mathbf{V} \in H(\Gamma)$$

×Ill posed problem !

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1^{st} *idea of regularization*

$$\mathbf{U} = \underset{\mathbf{V}\in H(\Gamma)}{\text{Argmin}} \left\{ \|\mathbf{V} - \Phi_d\|^2_{\Gamma_d} + c\|\mathbf{V} - \Phi\|^2_{\Gamma_i} \right\}$$

✓ Well posed optimization problem (control on the Γ_i part),
✓ Best agreement to the data (data relaxation),
× The solution depends on the choice of c and Φ !

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- independence to a regularization parameter,
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Iterative algorithm

$$\mathbf{U}^{k+1} = \underset{\mathbf{V}\in H(\Gamma)}{\text{Argmin}} \left\{ \|\mathbf{V} - \Phi_d\|^2_{\Gamma_d} + c\|\mathbf{V} - \mathbf{U}^k\|^2_{\Gamma_i} \right\}$$

✓ A sequence of well-posed optimization problems,
✓ Best agreement to the data (data relaxation),
✓ Independence of the solution with respect to c and Φ,
× No theoretical convergence result of the algorithm.

The fading regularization method

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Basic idea : Seeking among all solutions of the equilibrium equation in Ω, the one that fits the best the boundary conditions available on Γ_d, with :

- independence to a regularization parameter,
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The fading regularization method

$$\mathbf{U}^{k+1} = \underset{\mathbf{V}\in H(\Gamma)}{\text{Argmin}} \left\{ \|\mathbf{V} - \Phi_d\|^2_{\Gamma_d} + c\|\mathbf{V} - \mathbf{U}^k\|^2_{\Gamma} \right\}$$

✓ A sequence of well-posed optimization problems,
✓ Best agreement to the data (data relaxation),
✓ Independence of the solution with respect to c,
✓ Convergent algorithm.

→ At iteration k, there exists a unique minimum characterized by the optimality equation :

$$\langle \mathbf{U}^{k+1} - \Phi_d, \mathbf{V}\rangle_{\Gamma_d} + c\langle \mathbf{U}^{k+1} - \mathbf{U}^k, \mathbf{V}\rangle_{\Gamma} = 0 \quad \forall \mathbf{V} \in H(\Gamma)$$

Convergence of the continuous formulation

Theorem

Let Φ_d be the compatible Cauchy data associated with the compatible solution $\mathbf{U}_e \in \mathcal{H}(\Gamma)$. Then, the sequence $(\mathbf{U}^k)_{k\in\mathbb{N}}$ generated by the iterative algorithm verifies :

$$\mathbf{U}^k \to \Phi_d \quad \textit{in} \quad H(\Gamma_d) \quad \textit{strongly}$$
$$\mathbf{U}^k \rightharpoonup \mathbf{U}_e \quad \textit{in} \quad H(\Gamma) \quad \textit{weakly}$$

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Lemma

For all $n \in \mathbb{N}$, the sequence $(\mathbf{U}^k)_k$ generated by the iterative algorithm verifies:

$$\|\mathbf{U}^{n+1} - \mathbf{U}_e\|_\Gamma^2 + \sum_{k=0}^{n} \|\mathbf{U}^{k+1} - \mathbf{U}^k\|_\Gamma^2 + \frac{2}{c}\sum_{k=0}^{n}\|\mathbf{U}^{k+1} - \Phi_d\|_{\Gamma_d}^2 = \|\mathbf{U}^0 - \mathbf{U}_e\|_\Gamma^2$$

where $\mathbf{U}_e$ is the compatible solution of the Cauchy problem.

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- The strong convergence
  - The series $\sum_{k=0}^{n} \|\mathbf{U}^{k+1} - \Phi_d\|_{\Gamma_d}^2$ is bounded,

  $\rightarrow$ $\|\mathbf{U}^k - \Phi_d\|_{\Gamma_d}^2$ tends to 0,

  $\rightarrow$ $\mathbf{U}^k \underset{k \to +\infty}{\longrightarrow} \Phi_d$ on Γ_d.

Convergence of the continuous formulation

Lemma

For all $n \in \mathbb{N}$, the sequence $(\mathbf{U}^k)_k$ generated by the iterative algorithm verifies:

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where $\mathbf{U}_e$ is the compatible solution of the Cauchy problem.

- The weak convergence
  - Existence of a sub-sequence of $(\mathbf{U}^k)_k$ that is weakly convergent to $\mathbf{U}_e$
    - $(\|\mathbf{U}^k - \mathbf{U}_e\|_\Gamma^2)_k$ is bounded, hence $(\mathbf{U}^k)_k$ is bounded in $H(\Gamma)$

    $\rightarrow$ there exists a sub-sequence $(\mathbf{U}^\mu)_\mu$ of $(\mathbf{U}^k)_k$ such as :

$$\mathbf{U}^\mu \rightharpoonup \mathbf{U}_L \text{ in } H(\Gamma)$$

    - $\lim_{\mu \to +\infty} \|\mathbf{U}^\mu - \Phi_d\|_{\Gamma_d}^2 = 0$, hence $\lim_{\mu \to +\infty} \mathbf{U}^\mu = \Phi_d$
    - by uniquness of the limit on Γ_d: $\mathbf{U}_L|_{\Gamma_d} = \Phi_d$
    - by uniquness of the harmonic extension (**Holmgren's theorem**):

$$\mathbf{U}_L = \mathbf{U}_e \text{ on } \Gamma.$$

Convergence of the continuous formulation

Lemma

For all $n \in \mathbb{N}$, the sequence $(\mathbf{U}^k)_k$ generated by the iterative algorithm verifies:

$$\|\mathbf{U}^{n+1} - \mathbf{U}_e\|_\Gamma^2 + \sum_{k=0}^{n} \|\mathbf{U}^{k+1} - \mathbf{U}^k\|_\Gamma^2 + \frac{2}{c} \sum_{k=0}^{n} \|\mathbf{U}^{k+1} - \Phi_d\|_{\Gamma_d}^2 = \|\mathbf{U}^0 - \mathbf{U}_e\|_\Gamma^2$$

where $\mathbf{U}_e$ is the compatible solution of the Cauchy problem.

- The weak convergence
  - Existence of a sub-sequence of $(\mathbf{U}^k)_k$ that is weakly convergent to $\mathbf{U}_e$
  - Weak convergence of all the sequence $(\mathbf{U}^k)_k$ to $\mathbf{U}_e$ on Γ
    - $\rightarrow$ Proof by contradiction.

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where $\mathbf{U}_e$ is the compatible solution of the Cauchy problem.

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Remark

No equivalence of the harmonic extension (Holmgren's theorem) in finite dimension.
Convergence of the discrete formulation ?

Convergence of the discrete formulation

- $H_N(\Gamma)$: characterization space of $H(\Gamma)$ in finite dimension
- The discrete fading regularization method :

  Let $c > 0$ and $\mathbf{U}^0 \in H_N(\Gamma)$,

$$\left\{\begin{array}{l} \mathbf{U}_N^{k+1} \in H_N(\Gamma) \text{ such as :} \\ J_c^{k+1}(\mathbf{U}_N^{k+1}) \leq J_c^{k+1}(\mathbf{V}_N), \quad \forall \mathbf{V}_N \in H_N(\Gamma) \\ \text{where } J_c^{k+1}(\mathbf{V}_N) = \|\mathbf{V}_N|_{\Gamma_d} - \Phi_d\|_{\Gamma_d}^2 + c\|\mathbf{V}_N - \mathbf{U}_N^k\|_\Gamma^2 \text{ for } \mathbf{V}_N \in H_N(\Gamma) \end{array}\right. \tag{1}$$

  Φ_d: $4N_d-$vector of discrete data

- The elements of $H_N(\Gamma)$ that fit at best the N_d data elements
  - If $N_d > N$: a solution in the sense of least squares.
  - If $N_d \leq N$: an infinity of solutions, defined to an element of the kernel of the "discrete trace operator" on Γ_d :

$$Z_N(\Gamma) = \{\mathbf{U}_N \in H_N(\Gamma); \quad \mathbf{U}_N|_{\Gamma_d} = 0\}$$

$$Z_N^\perp(\Gamma) = \{\mathbf{U}_N \in H_N(\Gamma); \quad \langle \mathbf{U}_N, \mathbf{V}_N\rangle_\Gamma = 0 \quad \forall \mathbf{V}_N \in Z_N(\Gamma)\}$$

Convergence of the discrete formulation

- The discrete Cauchy problem can be defined as :

$$\begin{cases} \text{Find } \mathbf{U}_N^e \in Z_N^{\perp}(\Gamma) \quad \text{such as :} \\ \langle \mathbf{U}_N^e - \Phi_d, \mathbf{V}_N \rangle_{\Gamma_d} = 0, \quad \forall \mathbf{V}_N \in Z_N^{\perp}(\Gamma) \end{cases} \tag{2}$$

Theorem

If $\mathbf{U}_N^0 = 0$ *then the sequence* $(\mathbf{U}_N^k)_k$ *verifies the following properties :*

- $\mathbf{U}_N^k \in Z_N^{\perp}(\Gamma), \quad \forall k \geq 0,$
- *the sequence* $(\mathbf{U}_N^k)_k$ *converges to the unique solution* $\mathbf{U}_N^e$ *of the discrete Cauchy problem* (2).

$$\langle \mathbf{U}_N^{k+1} - \Phi_d, \mathbf{V}_N \rangle_{\Gamma_d} + c\langle \mathbf{U}_N^{k+1} - \mathbf{U}_N^k, \mathbf{V}_N \rangle_{\Gamma} = 0, \quad \forall \mathbf{V}_N \in H_N(\Gamma)$$

$$\mathbf{U}_N^k = \mathbf{z}_N^k + \mathbf{y}_N^k, \quad \forall \mathbf{z}_N^k \in Z_N(\Gamma), \quad \forall \mathbf{y}_N^k \in Z_N^{\perp}(\Gamma) \text{ and } \forall k \geq 0$$

$$\mathbf{V}_N = \mathbf{z}_N + \mathbf{y}_N, \quad \forall \mathbf{z}_N \in Z_N(\Gamma) \text{ and } \forall \mathbf{y}_N \in Z_N^{\perp}(\Gamma)$$

$$\Rightarrow \mathbf{z}_N^{k+1} = \mathbf{z}_N^k, \quad \forall k \geq 0$$

with the initialization $\mathbf{U}_N^0 = 0$, we obtain that $\mathbf{z}_N^k = \mathbf{z}_N^0 = 0, \quad \forall k \geq 0$

$$\Rightarrow \mathbf{U}_N^k \in Z_N^{\perp}(\Gamma), \quad \forall k \geq 0$$

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As in the continuous case, we have :

$$\lim_{k \to \infty} \|\mathbf{U}_N^{k+1} - \Phi_d\|_{\Gamma_d}^2 = 0$$

If the solution of the discrete Cauchy problem verifies compatibility hypothesis (i.e. $\mathbf{U}_N^e = \Phi_d$ on Γ_d)

$$\lim_{k \to \infty} \|\mathbf{U}_N^{k+1} - \mathbf{U}_N^e\|_{\Gamma_d}^2 = 0$$

By equivalence of finite dimensional norms, there exists $\alpha_N > 0$,

$$\|\mathbf{U}_N^{k+1} - \mathbf{U}_N^e\|_{\Gamma}^2 \leq \alpha_N \|\mathbf{U}_N^{k+1} - \mathbf{U}_N^e\|_{\Gamma_d}^2$$

Convergence of the discrete formulation

- The discrete Cauchy problem can be defined as :

$$\begin{cases} \text{Find } \mathbf{U}_N^e \in Z_N^\perp(\Gamma) \quad \text{such as :} \\ \langle \mathbf{U}_N^e - \Phi_d, \mathbf{V}_N \rangle_{\Gamma_d} = 0, \quad \forall \mathbf{V}_N \in Z_N^\perp(\Gamma) \end{cases} \tag{2}$$

Theorem

If $\mathbf{U}_N^0 = 0$ then the sequence $(\mathbf{U}_N^k)_k$ verifies the following properties :

- $\mathbf{U}_N^k \in Z_N^\perp(\Gamma), \quad \forall k \geq 0$,
- *the sequence $(\mathbf{U}_N^k)_k$ converges to the unique solution $\mathbf{U}_N^e$ of the discrete Cauchy problem* (2).

We can also prove that there exists $M > 0$ such that

$$\|\mathbf{U}_N^k - \mathbf{U}_N^e\|_\Gamma^2 \leq \left(\frac{c}{c + \alpha_N}\right)^k M, \quad \forall k \geq 0$$

$\rightarrow$ Contraction of the discrete algorithm.

Stopping criteria for the fading regularization algorithm

The sequences composed by :

- The scalar product

$$sp(\mathbf{U}^{k+1}) := \left\langle \mathbf{U}^{k+1} - \Phi_d, \mathbf{U}^{k+1} - \mathbf{U}^k \right\rangle_{\Gamma_d}$$

- The relaxation term

$$J_{\Gamma_d}^{k+1}(\mathbf{U}) = \|\mathbf{U}^{k+1}|_{\Gamma_d} - \Phi_d\|^2_{H(\Gamma_d)}$$

- The regularization term

$$J_{\Gamma}^{k+1}(\mathbf{U}) = c\|\mathbf{U}^{k+1} - \mathbf{U}^k\|^2_{H(\Gamma)}$$

- The value of the functional

$$J_c^{k+1}(\mathbf{U}) = \|\mathbf{U}^{k+1}|_{\Gamma_d} - \Phi_d\|^2_{H(\Gamma_d)} + c\|\mathbf{U}^{k+1} - \mathbf{U}^k\|^2_{H(\Gamma)}$$

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- The scalar product

$$sp(\mathbf{U}^{k+1}) := \left\langle \mathbf{U}^{k+1} - \Phi_d, \mathbf{U}^{k+1} - \mathbf{U}^k \right\rangle_{\Gamma_d} \leq 0, \quad \forall k > 0$$

- The relaxation term

$$J_{\Gamma_d}^{k+1}(\mathbf{U}) = \|\mathbf{U}^{k+1}|_{\Gamma_d} - \Phi_d\|^2_{H(\Gamma_d)}, \quad \searrow \quad \forall k > 0$$

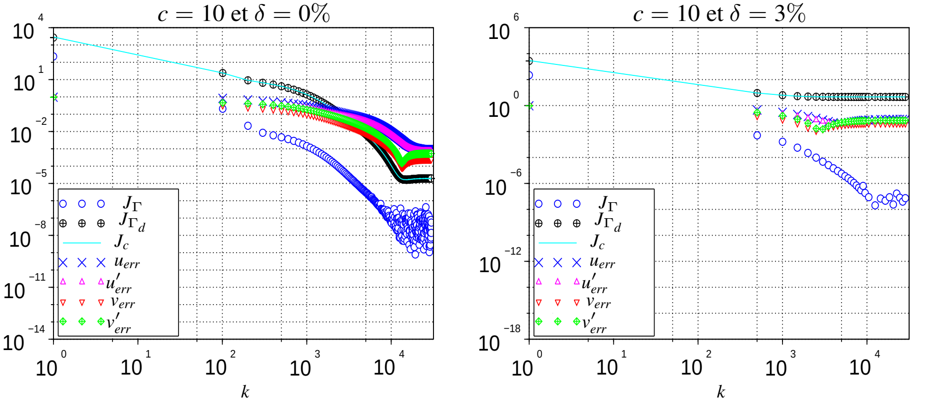
- The regularization term

$$J_{\Gamma}^{k+1}(\mathbf{U}) = c\|\mathbf{U}^{k+1} - \mathbf{U}^k\|^2_{H(\Gamma)}, \quad \searrow \quad \forall k > 0$$

- The value of the functional

$$J_c^{k+1}(\mathbf{U}) = \|\mathbf{U}^{k+1}|_{\Gamma_d} - \Phi_d\|^2_{H(\Gamma_d)} + c\|\mathbf{U}^{k+1} - \mathbf{U}^k\|^2_{H(\Gamma)}, \quad \searrow \quad \forall k > 0$$

Stopping criteria for the fading regularization algorithm



Proposition of a new stopping criterion

According to the lemma, for compatible data Φ_d, we have :

$$\underbrace{\sum_{j=0}^{k} \|\mathbf{U}^{j+1} - \mathbf{U}^{j}\|_{\Gamma}^{2} + \frac{2}{c} \sum_{j=0}^{k} \|\mathbf{U}^{j+1} - \Phi_d\|_{\Gamma_d}^{2}}_{S_d^{k+1}(\mathbf{U})} = \underbrace{\|\mathbf{U}_e\|_{\Gamma}^{2} - \|\mathbf{U}^{k+1} - \mathbf{U}_e\|_{\Gamma}^{2}}_{S_e^{k+1}(\mathbf{U})}$$

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According to the lemma, for compatible data Φ_d, we have :

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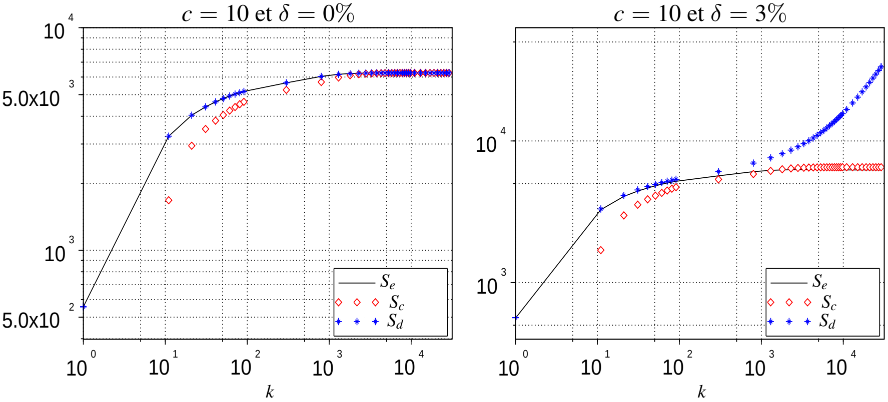
- For noisy data : $S_d^{k+1}(\mathbf{U}) \rightsquigarrow S_c^{k+1}(\mathbf{U})$

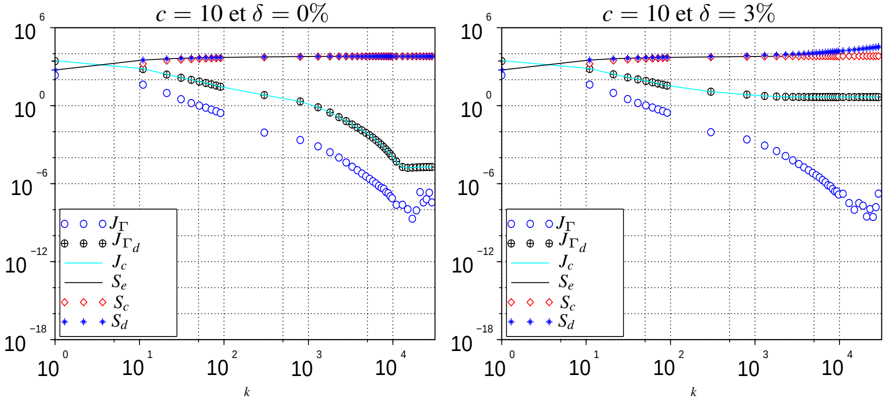
Idea : "*estimate the value of the accumulation due to noise* "

$$S_c^{k+1}(\mathbf{U}) := \sum_{j=0}^{k} \|\mathbf{U}^{j+1} - \mathbf{U}^{j}\|_{\Gamma}^{2} + \frac{2}{c}\left(\sum_{j=0}^{k} \|\mathbf{U}^{j+1} - \widetilde{\Phi_d}\|_{\Gamma_d}^{2} - (k+1)\|\mathbf{U}^{k+1} - \widetilde{\Phi_d}\|_{\Gamma_d}^{2}\right)$$

Proposition of a new stopping criterion



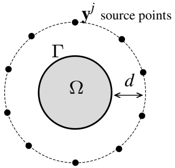
Proposition of a new stopping criterion



Numerical implementation using the method of fundamental solutions (MFS)

- Discretization of the space of solutions $H_N(\Gamma)$.
- Meshless method.
- Approximation by a linear combination of the fundamental solutions:

$$u(\mathbf{x}) \approx u^M(a, b, \underline{Y}; \mathbf{x}) = \sum_{j=1}^{M} a_j \mathscr{F}_1(\mathbf{x}, \mathbf{y}^j) + b_j \mathscr{F}_2(\mathbf{x}, \mathbf{y}^j), \quad \mathbf{x} \in \bar{\Omega}$$



$$\mathscr{F}_1(x, y) = -\frac{1}{2\pi} \ln r(x, y) \quad x \in \bar{\Omega}, \quad y \in \mathbb{R}^2 \backslash \bar{\Omega},$$

$$\mathscr{F}_2(x, y) = -\frac{1}{8\pi} r^2(x, y) \ln r(x, y) \quad x \in \bar{\Omega}, \quad y \in \mathbb{R}^2 \backslash \bar{\Omega}.$$

$$r(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Numerical implementation using the method of fundamental solutions (MFS)

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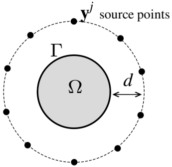
  - $\underline{\mathbf{X}}^T = (a_1, \dots, a_M, b_1, \dots, b_M)$ $2M-$vector of unknowns,
  - $\underline{\mathbf{U}}^T = (u, u_{,n}, v, v_{,n})$.



- Algebraic system

$$\mathcal{A}\underline{\mathbf{X}} = \underline{\mathbf{U}}$$

où $\mathcal{A} = \mathcal{A}(\mathbf{x}, \underline{Y}; \underline{n})$.

Resolution of the iterative algorithm using the MFS

- For $c > 0$ and $\mathbf{U}^0 = 0$, we define the sequence $\mathbf{X}^k$ that minimize J_c^k:

$$J_c^{k+1}(\mathbf{X}) = \|\mathcal{A}|_{\Gamma_d}\mathbf{X} - \Phi_d\|_{\Gamma_d}^2 + c\|\mathcal{A}\mathbf{X} - \mathcal{A}\mathbf{X}^k\|_{\Gamma}^2$$

- The iterative algorithm amounts to determining the sequence $(\mathbf{X}^k)_k$ such that :

$$\mathbf{X}^{k+1} = \underset{\mathbf{X}\in\mathbb{R}^{2M}}{\operatorname{Argmin}} \quad J_c^{k+1}(\mathbf{X})$$

Resolution of the iterative algorithm using the MFS

- For $c > 0$ and $\mathbf{U}^0 = 0$, we define the sequence $\mathbf{X}^k$ that minimize J_c^k:

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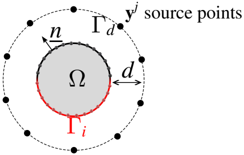
- The iterative algorithm amounts to determining the sequence $(\mathbf{X}^k)_k$ such that :

$$\mathbf{X}^{k+1} = \underset{\mathbf{X}\in\mathbb{R}^{2M}}{\text{Argmin}} \quad J_c^{k+1}(\mathbf{X})$$

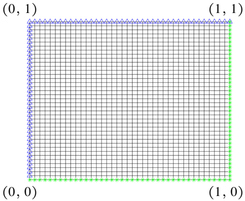
Inversible linear system

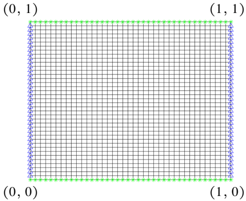
$$(\mathcal{A}^t|_{\Gamma_d}\mathcal{A}|_{\Gamma_d} + c\mathcal{A}^t\mathcal{A})\mathbf{X}^{k+1} = \mathcal{A}^t|_{\Gamma_d}\Phi_d + c\mathcal{A}^t\mathcal{A}\mathbf{X}^k$$

Numerical simulations



Unit disk





$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

Analytic solution

$\forall \mathbf{x} \in \bar{\Omega}$:

$u^{an}(\mathbf{x}) = \frac{1}{2} x_1 (\sin x_1 \cosh x_2 - \cos x_1 \sinh x_2)$,

$v^{an}(\mathbf{x}) = \Delta u^{an}(\mathbf{x}) = \cosh x_2 \cos x_1 + \sinh x_2 \sin x_1$,

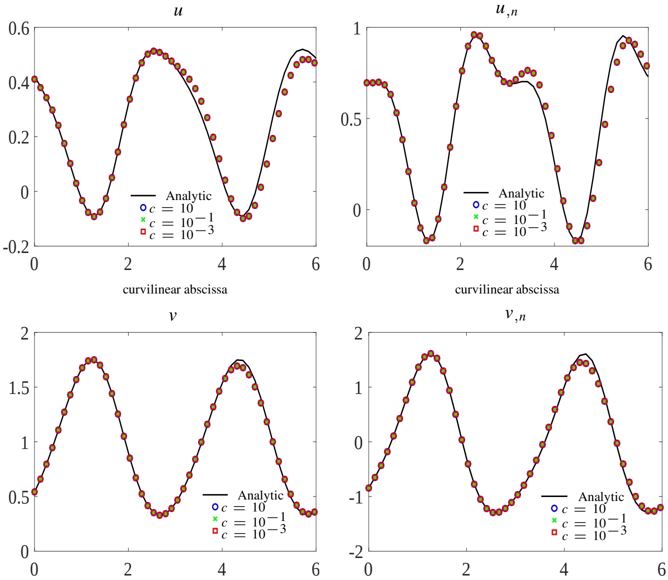
Noisy data

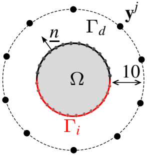
$\forall \mathbf{x} \in \Gamma_d$:

$$\begin{cases} \varphi_d(\mathbf{x}) = u^{an}(\mathbf{x}) + \delta \max(u^{an}(\mathbf{x}))\rho \\ \psi_d(\mathbf{x}) = u^{an}_{,n}(\mathbf{x}) + \delta \max(u^{an}_{,n}(\mathbf{x}))\rho \\ \mu_d(\mathbf{x}) = v^{an}(\mathbf{x}) + \delta \max(v^{an}(\mathbf{x}))\rho \\ \phi_d(\mathbf{x}) = v^{an}_{,n}(\mathbf{x}) + \delta \max(v^{an}_{,n}(\mathbf{x}))\rho \end{cases}$$

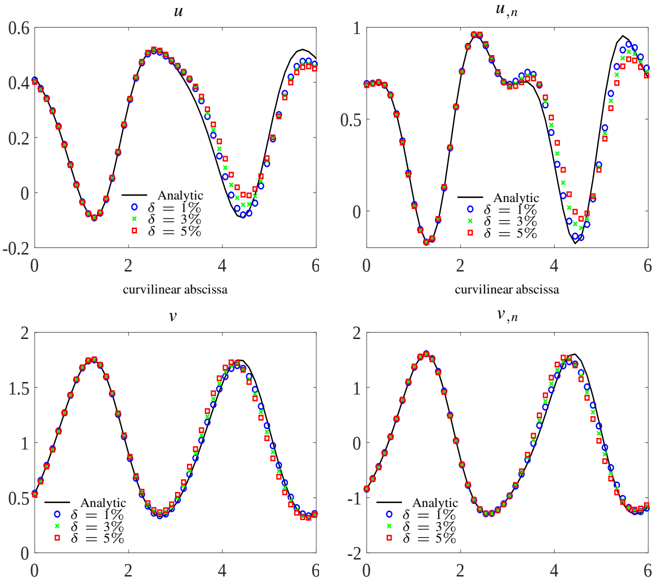
- δ : the percentage of noise level,
- ρ : a pseudo-random number in $[-1, 1]$.

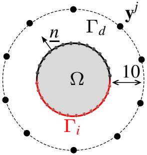
Independence towards the regularization parameter c



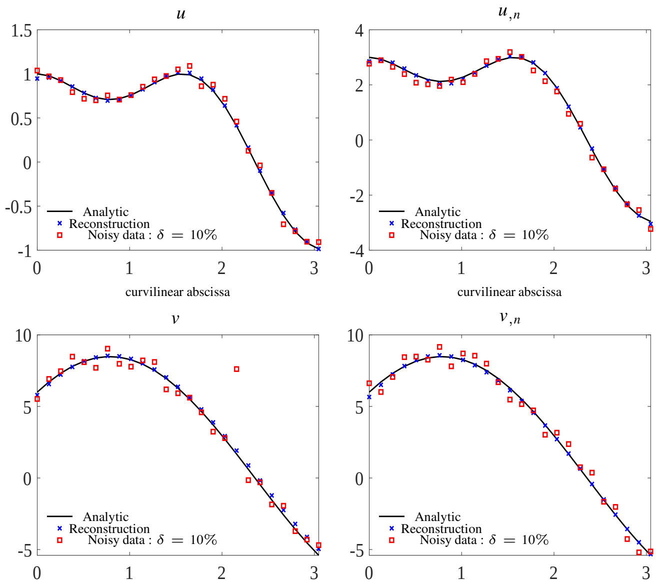


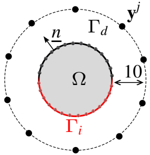
Stability towards noise level



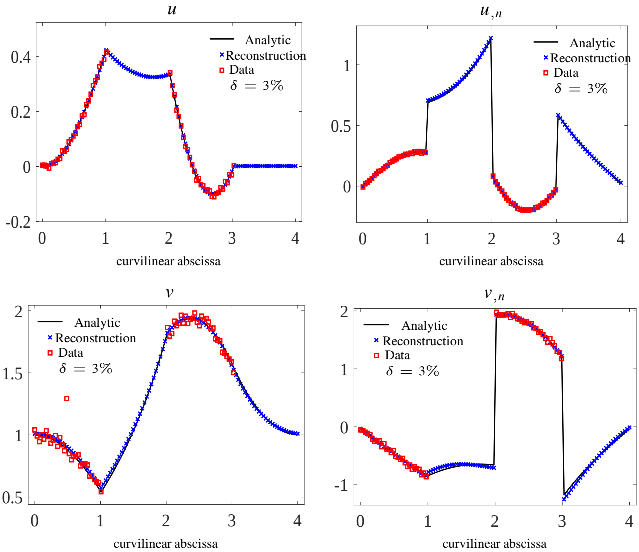


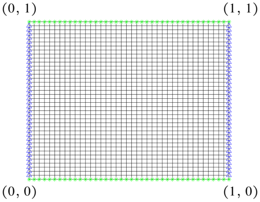
Ability to denoise the data





Reconstructions on the boundary of a square domain (noisy data located on two opposite sides)



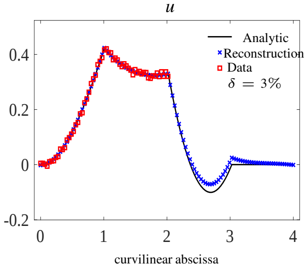


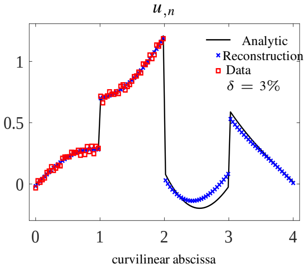
$\Omega = 40 \times 40$

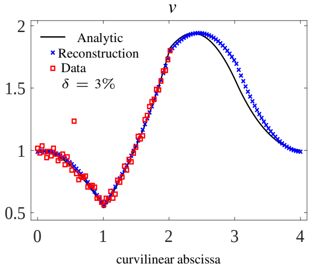
$\triangle$ Γ_i : unknowns

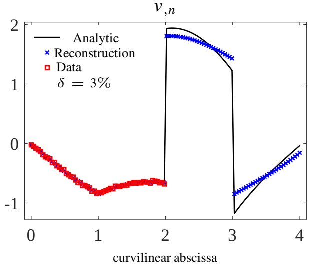
$*$ Γ_d : data

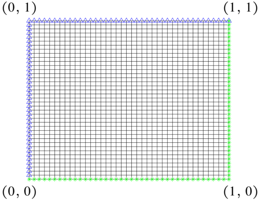
Reconstructions on the boundary of a square domain (noisy data located on two adjacent sides)











$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

Reformulation of the Cauchy biharmonic problem

Coupled formulation

$$\begin{cases} \Delta u = v, & \text{in } \Omega \\ u = \varphi_d, & \text{on } \Gamma_d \\ u_{,n} = \psi_d, & \text{on } \Gamma_d \\ \Delta v = 0, & \text{in } \Omega \\ v = \mu_d, & \text{on } \Gamma_d \\ v_{,n} = \phi_d, & \text{on } \Gamma_d \end{cases}$$

$$\begin{aligned} \mathbb{H}^h(\Gamma) = \{\mathbf{U} = (U, U', V, V') \in \mathbb{R}^N \times \mathbb{R}^{N'} \times \mathbb{R}^N \times \mathbb{R}^{N'} \mid \\ \varXi(V, V') \equiv KV + BV' = 0, \\ \varLambda(U, U', V) \equiv KU + BU' - DV = 0\}. \end{aligned}$$

Factorized formulation

$$\begin{cases} -\Delta v = 0, & \text{in } \Omega \\ v = \mu_d, & \text{on } \Gamma_d \\ v_{,n} = \phi_d & \text{on } \Gamma_d \end{cases}$$

$$\begin{cases} -\Delta u = v, & \text{in } \Omega \\ u = \varphi_d, & \text{on } \Gamma_d \\ u_{,n} = \psi_d & \text{on } \Gamma_d \end{cases}$$

$$\begin{aligned} \mathbb{H}_v^h(\Gamma) = \{\mathbf{V} = (V, V') \in \mathbb{R}^N \times \mathbb{R}^{N'} \mid \\ \varXi(V, V') \equiv KV + BV' = 0\}, \end{aligned}$$

and for $\mathbf{V} \in \mathbb{H}_v^h(\Gamma)$,

$$\begin{aligned} \mathbb{H}_u^h(\Gamma, V) = \{\mathbf{U} = (U, U') \in \mathbb{R}^N \times \mathbb{R}^{N'} \mid \\ \varLambda(U, U', V) \equiv KU + BU' - DV = 0\}. \end{aligned}$$

Numerical implementations using the FEM

Coupled formulation

Let $c > 0$ and $\mathbf{U}^0 = (0,0,0,0)$

$$\begin{cases} \mathbf{U}^{k+1} = \underset{\mathbb{R}^N \times \mathbb{R}^{N'} \times \mathbb{R}^N \times \mathbb{R}^{N'}}{\text{Argmin}} J_c^{k+1}(W,P,S,T) \\ \text{under the equality constraints : } \mathcal{E}(W,P,S,T) = 0, \end{cases}$$

$$\mathcal{E}(W,P,S,T) \equiv \begin{bmatrix} K & -D \\ 0 & K \end{bmatrix} \begin{bmatrix} W \\ S \end{bmatrix} + \begin{bmatrix} BP \\ BT \end{bmatrix}.$$

Factorized formulation

(a) Let $c_1 > 0$ and $\mathbf{V}^0 = (0,0)$

$$\begin{cases} \mathbf{V}^{k+1} = (V^{k+1}, {V'}^{k+1}) = \underset{\mathbb{R}^N \times \mathbb{R}^{N'}}{\text{Argmin}} J_{c_1}^{k+1}(V,V') \\ \text{under the equality constraints :} \\ \Xi(V,V') := KV + BV' = 0 \end{cases}$$

$\Rightarrow$ converges towards $\mathbf{V}_{\text{opt}} = (V_{\text{opt}}, {V'}_{\text{opt}})$

(b) Let $c_2 > 0$ and $\mathbf{U}^0 = (0,0)$

$$\begin{cases} \mathbf{U}^{k+1} = (U^{k+1}, {U'}^{k+1}) = \underset{\mathbb{R}^N \times \mathbb{R}^{N'}}{\text{Argmin}} J_{c_2}^{k+1}(U,U') \\ \text{under the equality constraints :} \\ \Lambda(U,U') := KU + BU' = DV_{\text{opt}} \end{cases}$$

Numerical implementations using the FEM

Coupled formulation

Let $c > 0$ and $\mathbf{U}^0 = (0,0,0,0)$

$$\begin{cases} \text{Find } (\mathbf{U}^{k+1}, \eta^{k+1}) \in \mathbb{R}^{N'} \times \mathbb{R}^{N'} \times \mathbb{R}^{N'} \times \mathbb{R}^{N'} \times \mathbb{R}^{N'} \\ \nabla J_c^{k+1}(\mathbf{U}^{k+1}) + (\eta^{k+1})^t \nabla \mathcal{E}(\mathbf{U}^{k+1}) = 0, \\ \mathcal{E}(\mathbf{U}^{k+1}) = 0. \end{cases}$$

$$\begin{bmatrix} \nabla J_c^{k+1} & \nabla \mathcal{E}^T \\ \mathcal{E} & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{U}}^{k+1} \\ \underline{\eta}^{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^k \\ 0 \end{bmatrix}$$

$$\mathbf{F}^k = \begin{bmatrix} M_{\Gamma_d} \underline{\varphi_d} + c\, M_\Gamma \underline{U}^k \\ M_{\Gamma_d} \underline{\psi_d} + c\, M_\Gamma \underline{U'}^k \\ M_{\Gamma_d} \underline{\mu_d} + c\, M_\Gamma \underline{V}^k \\ M_{\Gamma_d} \underline{\phi_d} + c\, M_\Gamma \underline{V'}^k \end{bmatrix}.$$

Factorized formulation

(a)Let $c_1 > 0$ and $\mathbf{V}^0 = (0,0)$

$$\begin{cases} \text{Find } (\mathbf{V}^{k+1}, \lambda^{k+1}) \in \mathbb{R}^{N'} \times \mathbb{R}^{N'} \times \mathbb{R}^{N'} \text{ such as} \\ \nabla J_{c_1}^{k+1}(\mathbf{V}^{k+1}) + (\lambda^{k+1})^t \nabla \Xi(\mathbf{V}^{k+1}) = 0, \\ \Xi(\mathbf{V}^{k+1}) = 0, \end{cases}$$

$$\begin{bmatrix} \nabla J_{c_1}^{k+1} & \nabla \Xi^T \\ \Xi & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{V}}^{k+1} \\ \underline{\lambda}^{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_v^k \\ 0 \end{bmatrix}$$

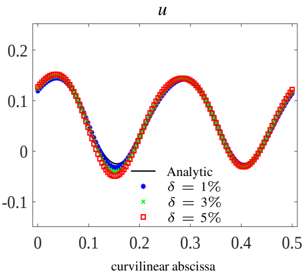
$\Rightarrow$ converges towards $\mathbf{V}_{\text{opt}} = (V_{\text{opt}}, V'_{\text{opt}})$

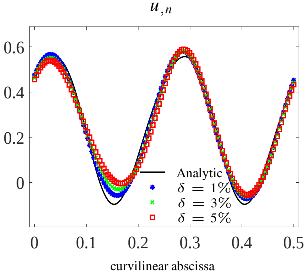
(b)Let $c_2 > 0$ and $\mathbf{U}^0 = (0,0)$

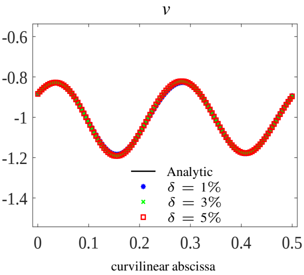
$$\begin{cases} \text{Find } (\mathbf{U}^{k+1}, \zeta^{k+1}) \in \mathbb{R}^{N'} \times \mathbb{R}^{N'} \times \mathbb{R}^{N'} \text{ such as} \\ \nabla J_{c_2}^{k+1}(\mathbf{U}^{k+1}) + (\zeta^{k+1})^t \nabla \Lambda(\mathbf{U}^{k+1}) = 0, \\ \Lambda(\mathbf{U}^{k+1}) = V_{\text{opt}}. \end{cases}$$

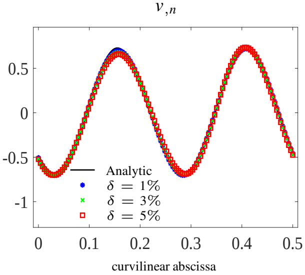
$$\begin{bmatrix} \nabla J_{c_2}^{k+1} & \nabla \Lambda^T \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{U}}^{k+1} \\ \underline{\zeta}^{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u^k \\ V_{\text{opt}} \end{bmatrix}$$

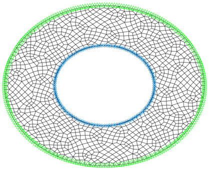
Reconstructions on the inner boundary of an annular domain (factorized formulation)





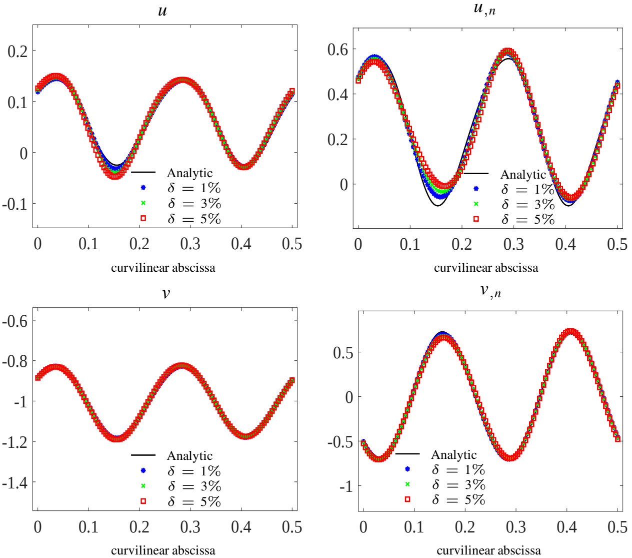


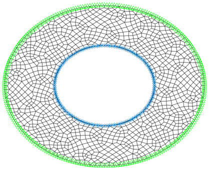




△ Γ_i : unknowns
∗ Γ_d : data

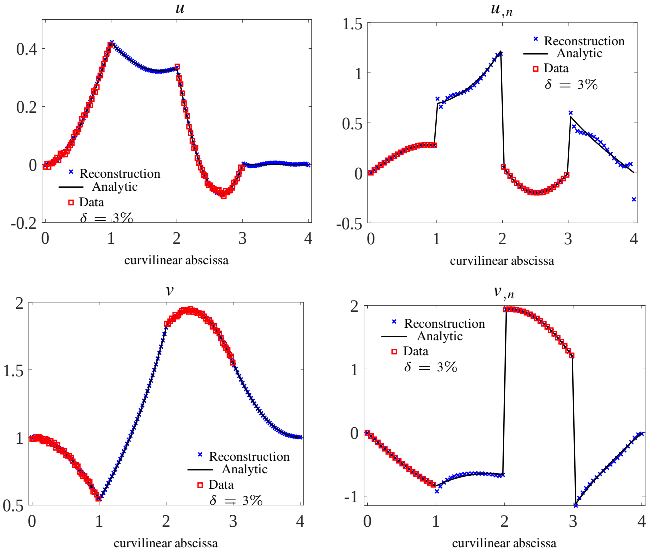
Reconstructions on the inner boundary of an annular domain (coupled formulation)

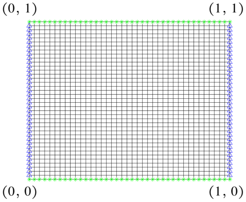




△ Γ_i : unknowns
∗ Γ_d : data

Reconstructions on the boundary of a square domain (noisy data located on two opposite sides)



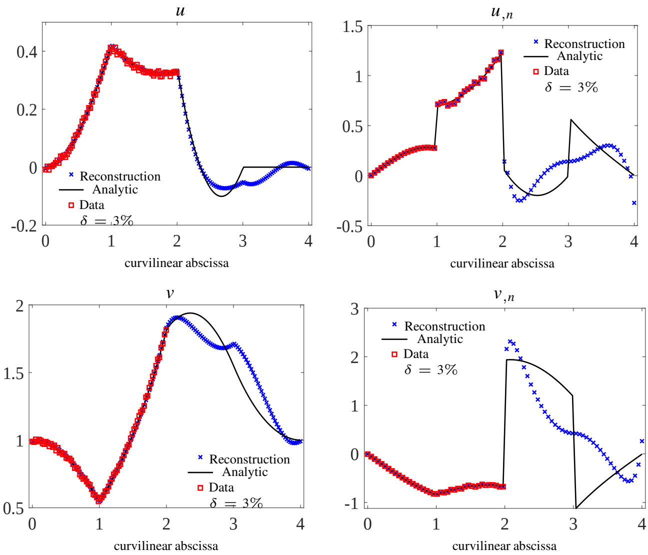


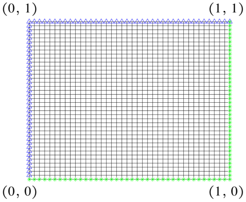
$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

Reconstructions on the boundary of a square domain (noisy data located on two adjacent sides)



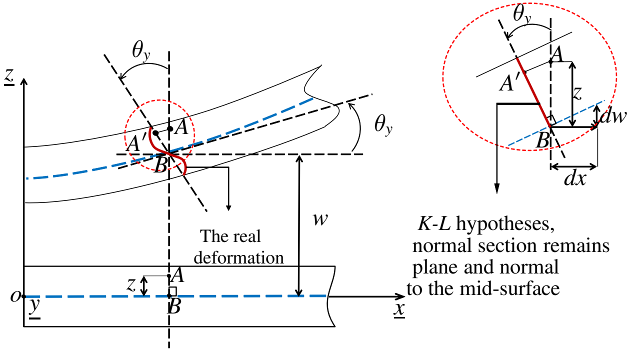


$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

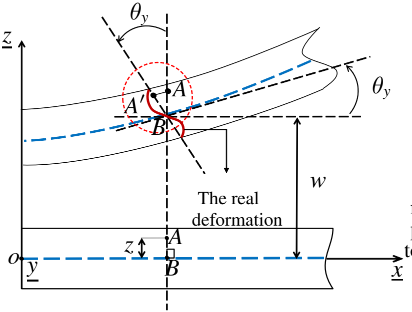
$*$ Γ_d : data

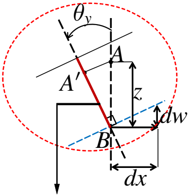
Kirchhoff-Love hypotheses



- *"Sections normal to the middle plane remain plane during deformation"*
- *"Sections normal to the middle plane remain normal to the middle plane during deformation"*

Kirchhoff-Love hypotheses

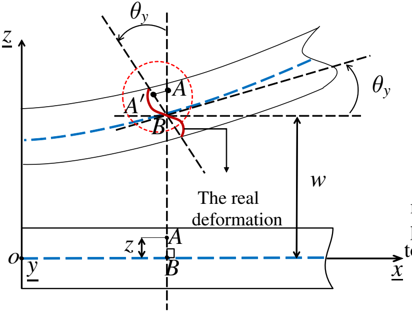


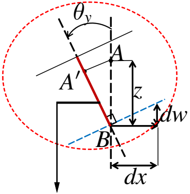


K-L hypotheses,
normal section remains
plane and normal
to the mid-surface

$$\begin{cases} \theta_x = \dfrac{\partial w}{\partial y} \\ \theta_y = -\dfrac{\partial w}{\partial x} \end{cases}$$

Kirchhoff-Love hypotheses





K-L hypotheses, normal section remains plane and normal to the mid-surface

- Variational formulation:

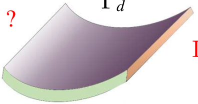
$$\int_\Omega \underbrace{\Big((\mathbf{L}\nabla)^t \mathbf{D}(\mathbf{L}\nabla) w\Big)}_{D\Delta^2 w} \delta w \, dxdy = \int_\Omega q(x,y)\delta\omega \quad dxdy$$

$$+ \int_\Gamma \left[\mathcal{M}_n \frac{\partial \delta w}{\partial n} - \mathcal{V}_n \delta w\right] \quad ds + \sum_i \delta w_i R_i$$

where $(\mathbf{L}\nabla) = \left[\frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} \quad 2\frac{\partial^2}{\partial x \partial y}\right]^t$ et $\mathbf{D}$ is the flexural rigidity of the plate.

Cauchy problem in thin plate theory

- Cauchy problem associated with the biharmonic equation with mechanical boundary conditions that relate to the thin plate bending problem



$$\begin{cases} \Delta^2 w = 0 & \text{in } \Omega \\ w = \varphi_d & \text{on } \Gamma_d \\ w_{,n} = \psi_d & \text{on } \Gamma_d \\ \mathcal{M}_n = \mathcal{M}_d & \text{on } \Gamma_d \\ \mathcal{V}_n = \mathcal{V}_d & \text{on } \Gamma_d \end{cases}$$

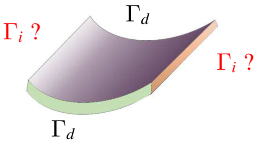
- The boundary conditions of the *Kirchhoff* thin plate theory amount to identifying the quantities w, $\frac{\partial w}{\partial n}$ and the forces :

$$\mathcal{M}_n = -D\left[\Delta w + (1-\nu)\left(2n_x n_y \frac{\partial^2 w}{\partial x \partial y} - n_y^2 \frac{\partial^2 w}{\partial x^2} - n_x^2 \frac{\partial^2 w}{\partial y^2}\right)\right]$$

$$\mathcal{V}_n = -D\left[\frac{\partial \Delta w}{\partial n} + (1-\nu)\frac{\partial}{\partial s}\left[n_x n_y \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right) + (n_x^2 - n_y^2)\frac{\partial^2 w}{\partial x \partial y}\right]\right]$$

Cauchy problem in thin plate theory

- Cauchy problem associated with the biharmonic equation with mechanical boundary conditions that relate to the thin plate bending problem



$$\begin{cases} \Delta^2 w = 0 & \text{in } \Omega \\ w = \varphi_d & \text{on } \Gamma_d \\ w_{,n} = \psi_d & \text{on } \Gamma_d \\ \mathcal{M}_n = \mathcal{M}_d & \text{on } \Gamma_d \\ \mathcal{V}_n = \mathcal{V}_d & \text{on } \Gamma_d \end{cases}$$

- The regularization functional becomes :

$$J_c^{k+1}(W) = \|w_{|\Gamma_d} - \phi_d\|^2_{H^{3/2}(\Gamma_d)} + \|w_{,n|\Gamma_d} - \mu_d\|^2_{H^{1/2}(\Gamma_d)} + \|\mathcal{M}_{n|\Gamma_d} - \mathcal{M}_d\|^2_{H^{-1/2}(\Gamma_d)}$$
$$+ \|\mathcal{V}_{n|\Gamma_d} - \mathcal{V}_d\|^2_{H^{-3/2}(\Gamma_d)} + c\Big(\|w - w^k\|_{H^{3/2}(\Gamma)} + \|w_{,n} - {w_{,n}}^k\|_{H^{1/2}(\Gamma)}$$
$$+ \|\mathcal{M}_n - \mathcal{M}_n^k\|_{H^{-1/2}(\Gamma)} + \|\mathcal{V}_n - \mathcal{V}_n^k\|_{H^{-3/2}(\Gamma)}\Big),$$
$$\forall W = (w, w_{,n}, \mathcal{M}_n, \mathcal{V}_n) \in \mathbf{H}(\Gamma).$$

where $c > 0$ and $\mathbf{H}(\Gamma)$ is the space of the compatible quadruplets.

Plate finite elements

Thin plates
Kirchhoff-Love theory

$\mathcal{C}^1$ regularity for the displacement w

Example of non-conforming elements :

- **-Adini 1960**
- **-Cheung 1968**
- **-Bazeley 1966**
- **-Melosh 1963**

Poor performance due to $\mathcal{C}^1$ regularity issue when rotation field interpolation comes from displacement field

Thick plates
Reissner-Mindlin theory

$\mathcal{C}^0$ regularity for w and independent interpolation for the rotation field

Example of thick plate elements :

- **-MITC4**
- **-DSQ**
- **-DKMQ**

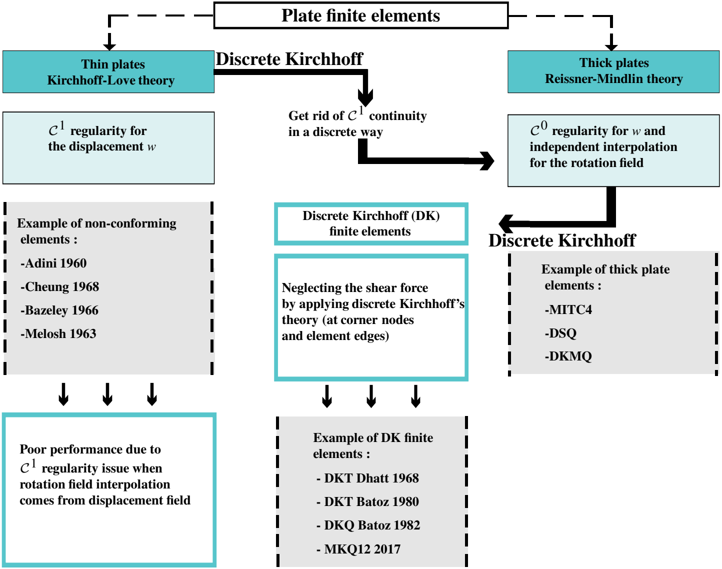
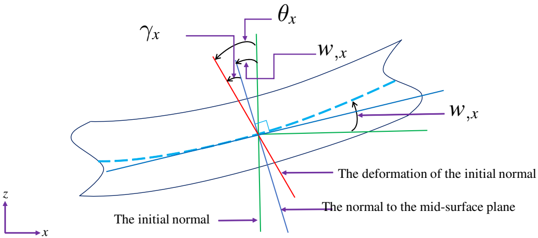


Plate finite elements
Thin plates
Kirchhoff-Love theory
Discrete Kirchhoff
Thick plates
Reissner-Mindlin theory
$\mathcal{C}^1$ regularity for the displacement w
Get rid of $\mathcal{C}^1$ continuity in a discrete way
$\mathcal{C}^0$ regularity for w and independent interpolation for the rotation field
Example of non-conforming elements :
-Adini 1960
-Cheung 1968
-Bazeley 1966
-Melosh 1963
Discrete Kirchhoff (DK) finite elements
Discrete Kirchhoff
Example of thick plate elements :
-MITC4
-DSQ
-DKMQ
Neglecting the shear force by applying discrete Kirchhoff's theory (at corner nodes and element edges)
Poor performance due to $\mathcal{C}^1$ regularity issue when rotation field interpolation comes from displacement field
Example of DK finite elements :
- DKT Dhatt 1968
- DKT Batoz 1980
- DKQ Batoz 1982
- MKQ12 2017

DK (Discrete Kirchhoff) finite elements

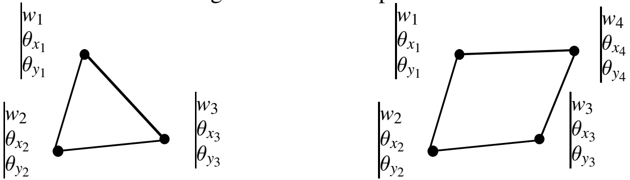
- **Thick plate finite element** : Including shear deformation $\theta_s = \gamma_s + \frac{\partial w}{\partial s}$



- Independent discretization of the displacement and the rotation field :

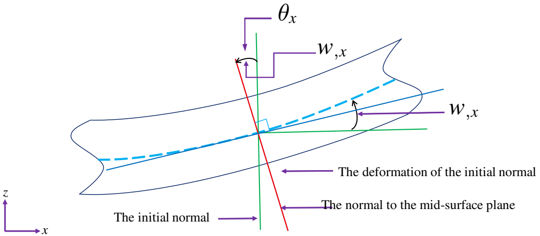
$$w = \sum_i N_i w_i \quad \theta_x = \sum_i N_i \theta_{x_i} \quad \theta_y = \sum_i N_i \theta_{y_i}$$

- Finite element with 3 degrees of freedom per node :



DK (Discrete Kirchhoff) finite elements

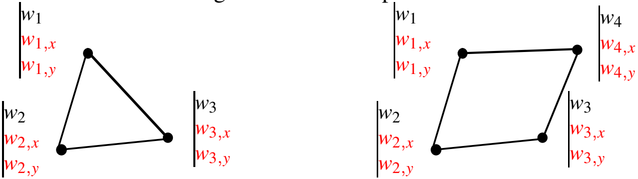
- **DK finite element** : *Kirchhoff* hypotheses $\gamma_s = 0 \Rightarrow \theta_s = \frac{\partial w}{\partial s}$



- Independent discretization of the displacement and the rotation field :

$$w = \sum_i N_i w_i \quad \theta_x = \sum_i N_i \theta_{x_i} \quad \theta_y = \sum_i N_i \theta_{y_i} \text{ such that } \theta_{s_i} = \frac{\partial w}{\partial s}\Big|_i$$

- Finite element with 3 degrees of freedom per node :



Numerical implementation of the iterative algorithm

- Interpolation of the displacement vector :

$$\underline{w}^e = \underline{\mathbf{N}}\, \underline{d}^e, \quad \underline{d}^{e_i} = \begin{bmatrix} w_i \\ \theta_{x_i} \\ \theta_{y_i} \end{bmatrix} = \begin{bmatrix} w_i \\ w_{,y_i} \\ -w_{,x_i} \end{bmatrix}.$$

- Interpolation the strain vector :

$$(\mathbf{L}\nabla)\underline{w}^e = \begin{bmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \theta_{x,y} + \theta_{y,x} \end{bmatrix} = \underline{B}^e\ \underline{d}^e$$

- Finite element formulation :

$$\Big(\int_\Omega \underline{B}^t \underline{D}\, \underline{B}\, d\Omega \Big)\underline{d} = \int_\Gamma \Big[-\underline{\mathbf{N}}^t_{,n}\mathcal{M}_n + \underline{\mathbf{N}}^t\mathcal{V}_n \Big]\, ds$$

$$\mathbf{K}\underline{d} = \underbrace{\Big[-\int_\Gamma \mathbf{N}^t_{,n} ds \quad \int_\Gamma \mathbf{N}^t ds \Big]}_{\equiv \mathbf{F}} \underbrace{\begin{bmatrix} \underline{\mathcal{M}}_n \\ \underline{\mathcal{V}}_n \end{bmatrix}}_{\equiv \underline{b}}$$

$$\mathcal{E}(\underline{\mathbf{V}}) := \mathbf{K}\underline{d} - \mathbf{F}\ \underline{b} = 0, \text{tel que } \underline{\mathbf{V}} = (\underline{d}, \underline{\mathcal{M}}_n, \underline{\mathcal{V}}_n)$$

Numerical implementation of the iterative algorithm

- The fading regularization algorithm :

$$\left\| \begin{array}{l} \underline{\mathbf{V}}^{k+1} = \underset{\underline{\mathbf{V}} \in \mathbb{R}^{5N}}{\text{Argmin}}\ J_c^{k+1}(\underline{\mathbf{V}}) \\ \text{with } \underline{\mathbf{V}} = (\underline{d}, \underline{\mathcal{M}}_n, \underline{\mathcal{V}}_n) = (\underline{W}, \underline{\theta}_{,x}, \underline{\theta}_{,y}, \underline{\mathcal{M}}_n, \underline{\mathcal{V}}_n) \\ \text{under the equality constraints } \mathcal{E}(\underline{\mathbf{V}}) = 0 \end{array} \right.$$

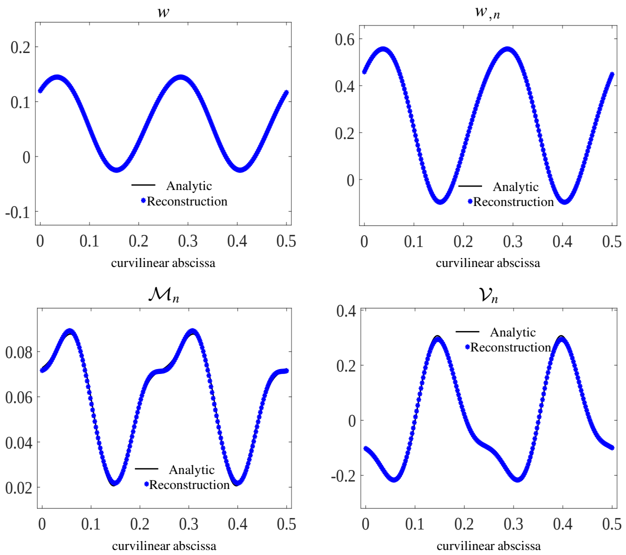
- The functional to be optimized :

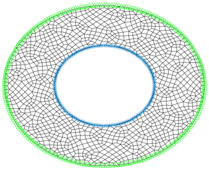
$$\begin{aligned} J_c^{k+1}(\underline{\mathbf{V}}) = & \|\underline{W}_{|\Gamma_d} - \underline{\phi}_d\|^2_{L^2(\Gamma_d)} + \|\underline{n}_y\underline{\theta}_{,x} + \underline{n}_x\underline{\theta}_{,y}{}_{|\Gamma_d} - \mu_d\|^2_{L^2(\Gamma_d)} \\ & + \|\underline{\mathcal{M}}_{n|\Gamma_d} - \underline{\mathcal{M}}_d\|^2_{L^2(\Gamma_d)} + \|\underline{\mathcal{V}}_{n|\Gamma_d} - \underline{\mathcal{V}}_d\|^2_{L^2(\Gamma_d)} + c\Big(\|\underline{W} - \underline{W}^k\|^2_{L^2(\Gamma)} \\ & + \|\underline{\theta}_{,x} - \underline{\theta}_{,x}{}^k\|^2_{L^2(\Gamma)} + \|\underline{\theta}_{,y} - \underline{\theta}_{,y}{}^k\|^2_{L^2(\Gamma)} + \|\underline{\mathcal{M}}_n - \underline{\mathcal{M}}_n^k\|^2_{L^2(\Gamma)} + \|\underline{\mathcal{V}}_n - \underline{\mathcal{V}}_n^k\|^2_{L^2(\Gamma)}\Big) \end{aligned}$$

- Resolution of the linear system :

$$\begin{bmatrix} \nabla J_c^{k+1} & \nabla \mathcal{E}^T \\ \mathcal{E} & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{V}}^{k+1} \\ \underline{\eta}^{k+1} \end{bmatrix} = \begin{bmatrix} \mathcal{S}^k \\ \underline{0} \end{bmatrix}.$$

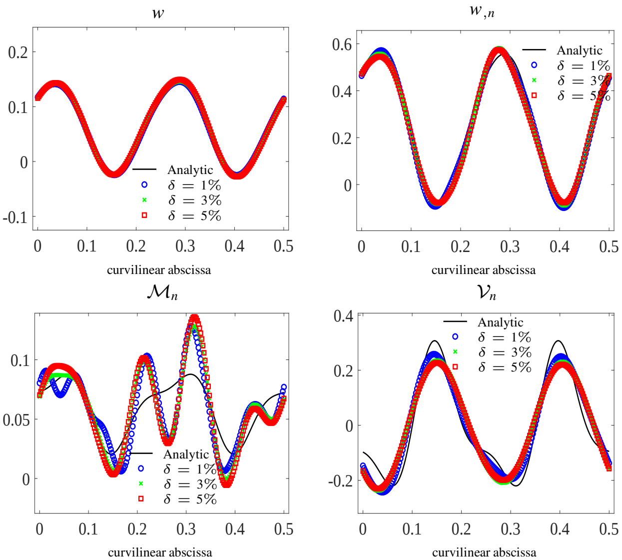
Reconstructions on the inner boundary of an annular domain (compatible data)

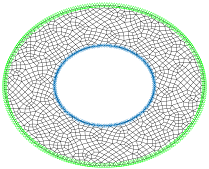




△ Γ_i : unknowns
∗ Γ_d : data

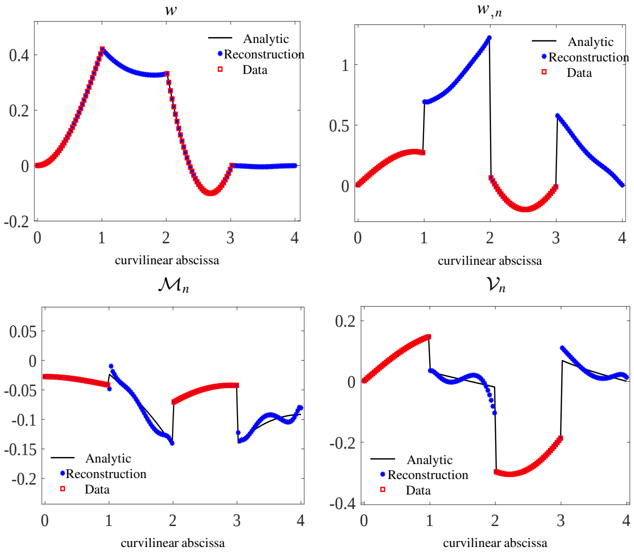
Reconstructions on the inner boundary of an annular domain (noisy data)

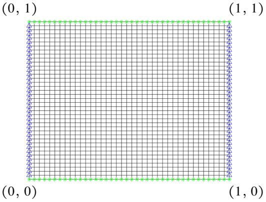




$\triangle$ Γ_i : unknowns
$*$ Γ_d : data

Reconstructions on the boundary of a square domain (compatible data located on two opposite sides)



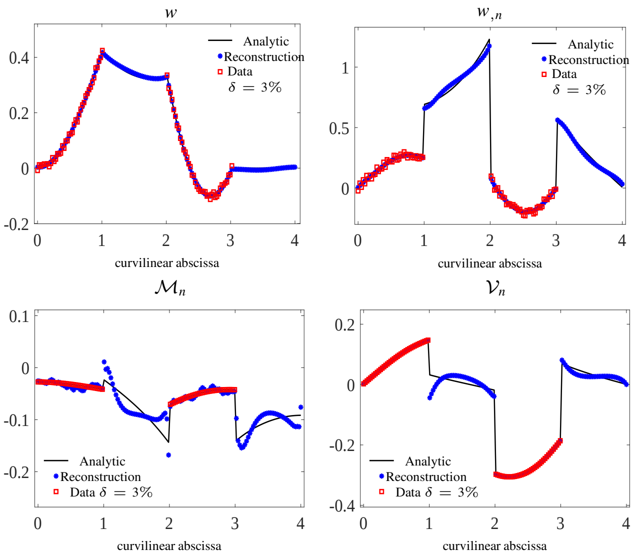


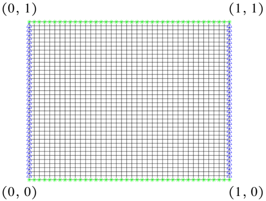
$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

Reconstructions on the boundary of a square domain (noisy data located on two opposite sides)



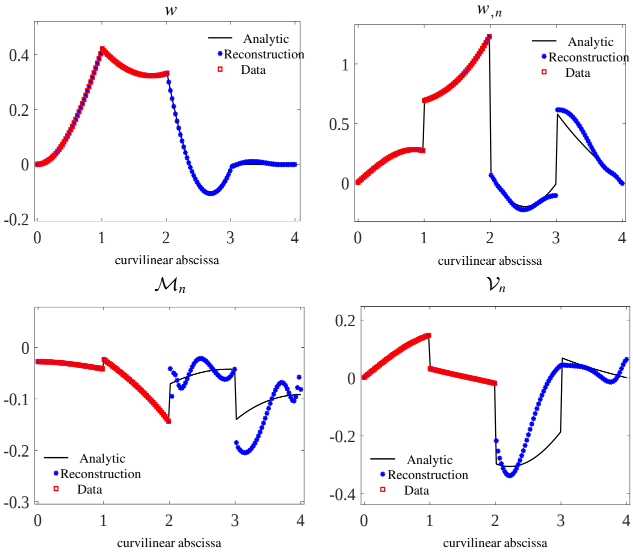


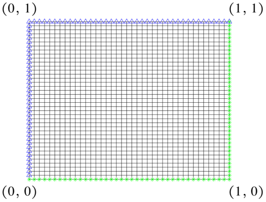
$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

Reconstructions on the boundary of a square domain (compatible data located on two adjacent sides)



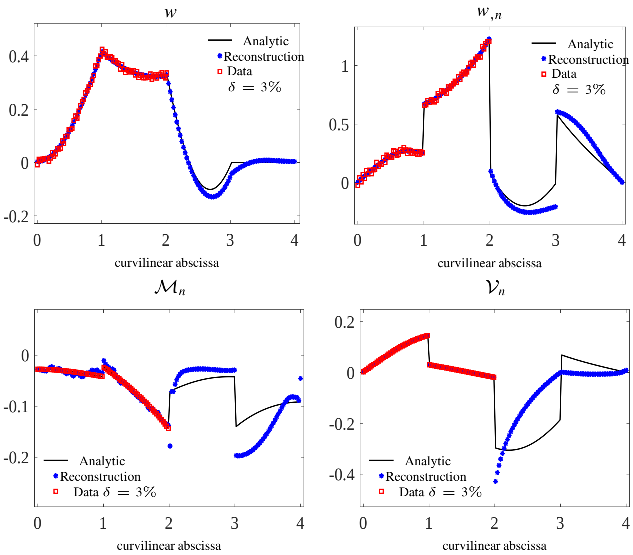


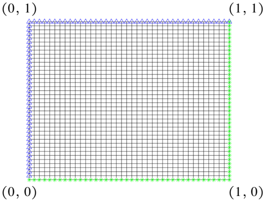
$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

Reconstructions on the boundary of a square domain (noisy data located on two adjacent sides)





$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

The Cauchy problem associated with the Laplace equation

- The Cauchy problem associated with the Laplace equation

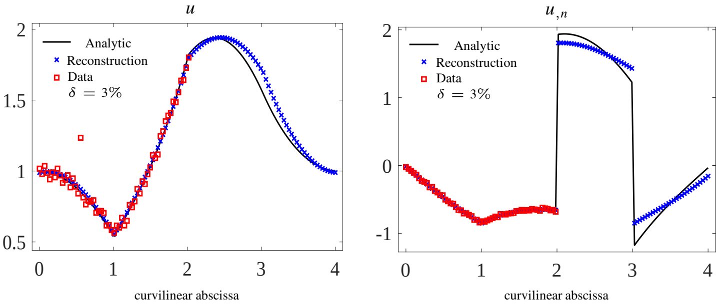
$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega \\ u = \phi_d \quad \text{on } \Gamma_d \\ u_{,n} = \mu_d \quad \text{on } \Gamma_d \end{cases}$$

The Cauchy problem associated with the Laplace equation

- The Cauchy problem associated with the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi_d & \text{on } \Gamma_d \\ u_{,n} = \mu_d & \text{on } \Gamma_d \end{cases}$$

- Results obtained using the method of fundamental solutions:

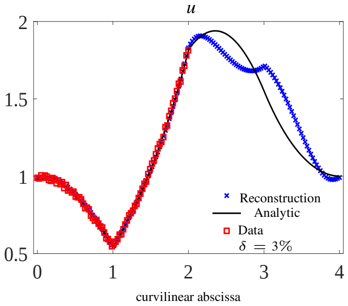


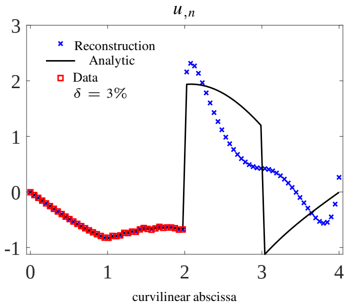
The Cauchy problem associated with the Laplace equation

- The Cauchy problem associated with the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi_d & \text{on } \Gamma_d \\ u_{,n} = \mu_d & \text{on } \Gamma_d \end{cases}$$

- Results obtained by the finite element method:



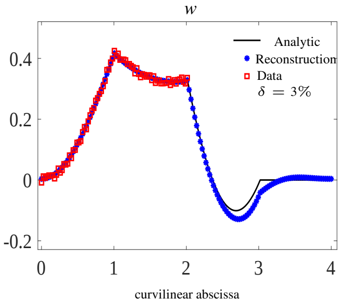


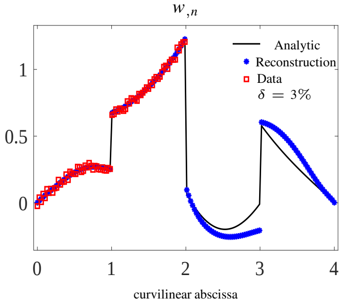
The Cauchy problem associated with the Laplace equation

- The Cauchy problem associated with the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi_d & \text{on } \Gamma_d \\ u_{,n} = \mu_d & \text{on } \Gamma_d \end{cases}$$

- Results obtained by plate finite elements (DK):





Adaptation of Melosh finite element for the Laplacian

- Cubic interpolation for displacement

$$\begin{aligned} u =& \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y \\ &+ \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3 \\ u =& \mathbf{P}\underline{\alpha} \end{aligned}$$

- The degrees of freedom vector can be derived as

$$\underline{d} = \left\| \begin{aligned} u_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 x_i^2 + \dots \\ (\frac{\partial u}{\partial y})_i &= \alpha_3 + \alpha_5 x_i + \dots \\ -(\frac{\partial u}{\partial x})_i &= -\alpha_2 - \alpha_5 y_i + \dots \end{aligned} \right.$$

$$\underline{d} = \mathbf{C}\underline{\alpha} \Rightarrow \underline{\alpha} = \mathbf{C}^{-1}\underline{d}$$

- The vector of interpolation functions

$$u = \mathbf{P}\underline{\alpha} = \underbrace{\mathbf{P}\mathbf{C}^{-1}}_{\mathbf{N}}\underline{d} = \mathbf{N}\underline{d}.$$

- Finite element formulation associated with Laplace's equation

$$\int_\Omega \nabla u \nabla v d\Omega = \int_\Gamma \frac{\partial u}{\partial n} v d\sigma, \quad \forall v \in H_0^1(\Omega)$$

$$\Big(\int_\Omega \nabla \mathbf{N}^t \nabla \mathbf{N}\, d\Omega\Big)\underline{d} = \Big(\int_\Gamma \mathbf{N}^t \mathbf{N}_{,n} d\sigma\Big)\underline{d}$$

$$\mathbf{K}\underline{d} = \mathbf{F}\underline{d}$$

New regularization strategy

- The fading regularization algorithm :
  Let $c > 0$ and $\mathbf{U}^0 \in \mathbf{H}(\Gamma)$,

$$\begin{cases} \text{Find } \mathbf{U}^{k+1} = (u^{k+1}, u^{k+1}_{,x}, u^{k+1}_{,y}) \in \mathbf{H}(\Gamma) \text{ tel que} \\ J^{k+1}_c(\mathbf{U}^{k+1}) \leq J^{k+1}_c(\mathbf{V}), \quad \forall \mathbf{V} = (v, v_{,x}, v_{,y}) \in \mathbf{H}(\Gamma), \quad \forall k \geq 0, \\ \text{où } J^{k+1}_c(\mathbf{V}) = \|v_{|\Gamma_d} - \phi_d\|^2_{H^1(\Gamma_d)} + \|(n_x v_{,x|\Gamma_d} + n_y v_{,y|\Gamma_d}) - \mu_d\|^2_{L^2(\Gamma_d)} \\ \qquad\qquad + c\|\mathbf{V} - \mathbf{V}^k\|^2_{\mathbf{H}(\Gamma)} \end{cases}$$

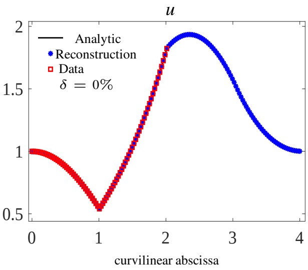
- The discrete fading regularization algorithm :

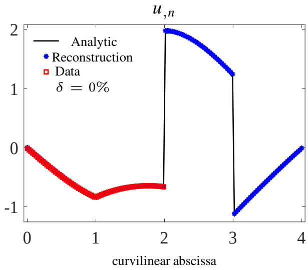
$$\left\| \begin{array}{l} \underset{\underline{\mathbf{U}} \in \mathbb{R}^{3N}}{\text{Argmin}}\, J^{k+1}_c(\underline{\mathbf{U}}) \\ \text{such as } \big(\mathbf{K} - \mathbf{F}\big)\underline{\mathbf{U}} = 0 \end{array} \right.$$

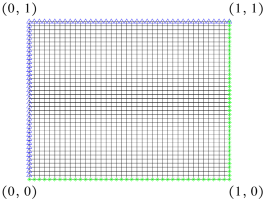
where

$$J^{k+1}_c(\underline{\mathbf{V}}) = \|\underline{V}_{|\Gamma_d} - \underline{\phi}_d\|^2_{L^2(\Gamma_d)} + \|(\underline{n}_x \underline{V}_{,x|\Gamma_d} + \underline{n}_y \underline{V}_{,y|\Gamma_d}) - \underline{\mu}_d\|^2_{L^2(\Gamma_d)}$$
$$+ c\Big(\|\underline{V} - \underline{U}^k\|_{L^2(\Gamma)} + \|\underline{V}_{,x} - \underline{U}^k_{,x}\|_{L^2(\Gamma)} + \|\underline{V}_{,y} - \underline{U}^k_{,y}\|_{L^2(\Gamma)}\Big),$$
$$\forall \underline{\mathbf{V}} = (\underline{V}, \underline{V}_{,x}, \underline{V}_{,y}) \in \mathbb{R}^{3N}.$$

Reconstructions on the boundary of a square domain (data located on two adjacent sides)





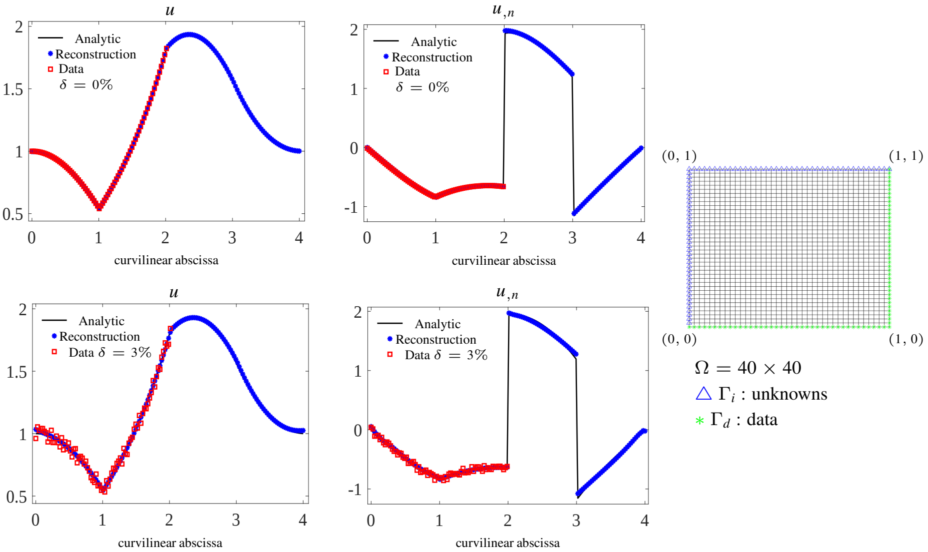


$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

Reconstructions on the boundary of a square domain (data located on two adjacent sides)



$\Omega = 40 \times 40$

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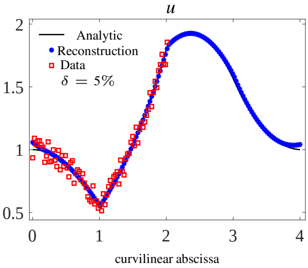
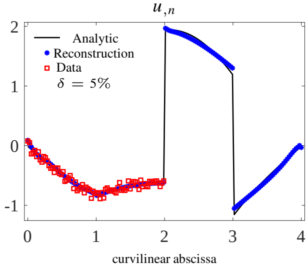
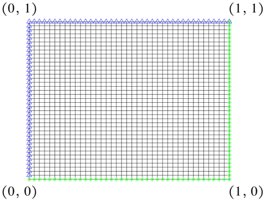
Reconstructions on the boundary of a square domain (data located on two adjacent sides)





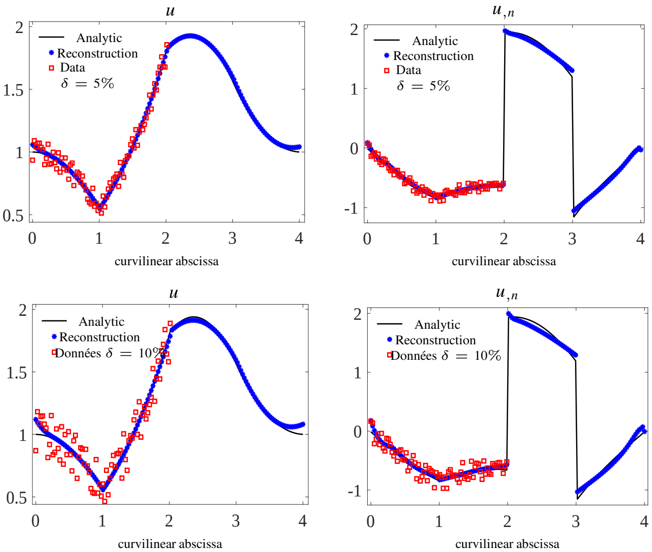


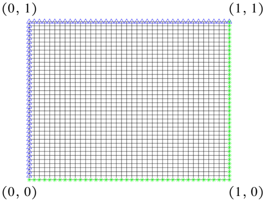
$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

Reconstructions on the boundary of a square domain (data located on two adjacent sides)





$\Omega = 40 \times 40$

$\triangle$ Γ_i : unknowns

$*$ Γ_d : data

1. The biharmonic Cauchy problem
2. Cauchy problem in thin plate theory
3. Plate finite element for second order Cauchy problem
4. **Conclusion and Outlooks**

Conclusion and Outlooks

- The fading regularization method :
  - converges to a solution of the equilibrium equation
  - robust (stable) towards noise
  - able to denoise data
- Numeric implementation of the fading regularization algorithm using MFS, FEM and Discrete Kirchhoff finite elements
- Reconstruction of the boundary conditions of the biharmonic Cauchy problem and of the Cauchy problem in thin plate theory for smooth and non-smooth domains
- Outlooks :
  - From a numerical point of view
    - Use of MFS for the Cauchy problem in thin plate theory
    - Use of other types of plate finite elements that ensure $\mathcal{C}^1$ continuity (idea : adding the cross derivative as nodal parameter (Bogner or Bazeley elements))
  - Related to mechanics
    - Data completion problems in thin plate theory (identification of fields and/or boundary conditions, identification of defects, etc...)
    - Use of experimental and real data

Thank you

Feel free to add me to your contacts!

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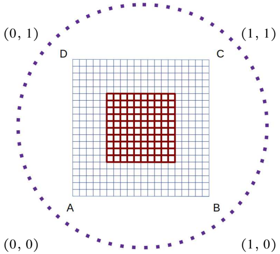
Inria researcher (Postdoc position)
IDEFIX group, Inria, ENSTA Paris, Institut Polytechnique de Paris

Data completion problem using interior measurements

$$\begin{cases} \Delta^2 u = 0 & \forall x \in \Omega \\ u = \phi_d & \forall x \in \Omega_d \end{cases}$$



Ω

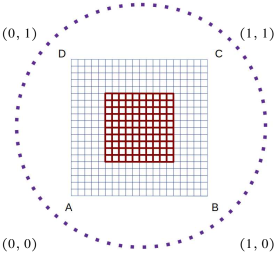
□ Ω_d : data

□ Ω_i : unknowns

• source points

Data completion problem using interior measurements by the MFS

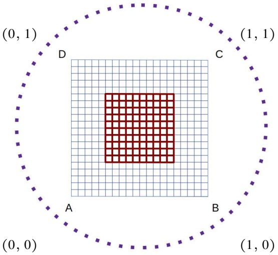
$$\begin{cases} \Delta^2 u = 0 & \forall x \in \Omega \\ u = \phi_d & \forall x \in \Omega_d \end{cases}$$



$$u(\mathbf{x}) \approx u^M(a, b, \underline{\mathbf{y}}; \mathbf{x}) = \sum_{j=1}^{M} a_j \mathscr{F}_1(\mathbf{x}, \mathbf{y}^j) + b_j \mathscr{F}_2(\mathbf{x}, \mathbf{y}^j), \quad \mathbf{x} \in \bar{\Omega} \tag{3}$$

Data completion problem using interior measurements by the MFS

$$\begin{cases} \Delta^2 u = 0 & \forall x \in \Omega \\ u = \phi_d & \forall x \in \Omega_d \end{cases}$$



$$u(\mathbf{x}) \approx u^M(a, b, \underline{\mathbf{y}}; \mathbf{x}) = \sum_{j=1}^{M} a_j \mathscr{F}_1(\mathbf{x}, \mathbf{y}^j) + b_j \mathscr{F}_2(\mathbf{x}, \mathbf{y}^j), \quad \mathbf{x} \in \bar{\Omega} \tag{3}$$

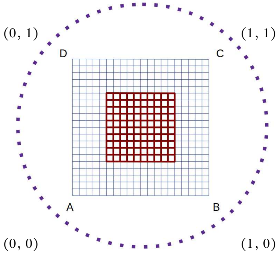
$$u_{,n}(\underline{\mathbf{x}}) \approx \frac{\partial u}{\partial n}(a, b, \underline{\mathbf{y}}, \underline{\mathbf{n}}; \underline{\mathbf{x}}) = \sum_{j=1}^{M} a_j \mathscr{F}'_1(\underline{\mathbf{x}}, \mathbf{y}^j; \underline{\mathbf{n}}) + b_j \mathscr{F}'_2(\underline{\mathbf{x}}, \mathbf{y}^j; \underline{\mathbf{n}}), \quad \underline{\mathbf{x}} \in \Gamma,$$

$$v(\underline{\mathbf{x}}) \approx \Delta u(a, b, \underline{\mathbf{y}}; \underline{\mathbf{x}}) = \sum_{j=1}^{M} b_j \mathscr{G}_2(\underline{\mathbf{x}}, \mathbf{y}^j), \quad \underline{\mathbf{x}} \in \Gamma, \tag{4}$$

$$v_{,n}(\underline{\mathbf{x}}) \approx \frac{\partial v}{\partial n}(a, b, \underline{\mathbf{y}}, \underline{\mathbf{n}}; \underline{\mathbf{x}}) = \sum_{j=1}^{M} b_j \mathscr{G}'_2(\underline{\mathbf{x}}, \mathbf{y}^j), \quad \underline{\mathbf{x}} \in \Gamma.$$

Data completion problem using interior measurements by the MFS

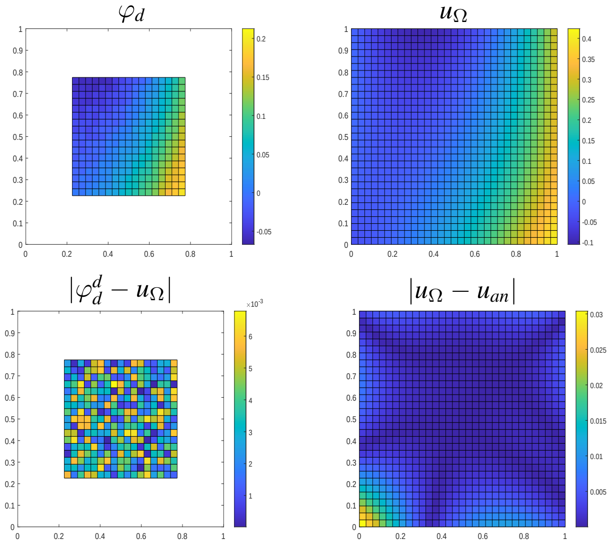
$$\begin{cases} \Delta^2 u = 0 & \forall x \in \Omega \\ u = \phi_d & \forall x \in \Omega_d \end{cases}$$

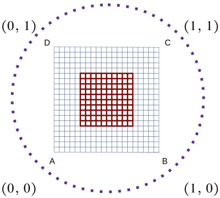


Given $c > 0$ and $u^0 \in H(\Omega)$,

$$\begin{cases} \text{Find } u^{k+1} \in H(\Omega) \quad \text{such as :} \\ J_c^k(u^{k+1}) \leq J_c^k(w), \quad \forall w \in H(\Omega) \\ J_c^k(w) = \|w_{|\Omega_d} - \varphi_d\|^2_{H(\Omega_d)} + c\|w - u^k\|^2_{H(\Omega)} \end{cases} \tag{3}$$

Reconstructions of the solution u inside a square domain by the MFS from partial interior noisy data measurements ($\delta = 3\%$)





$\Omega = 30 \times 30$

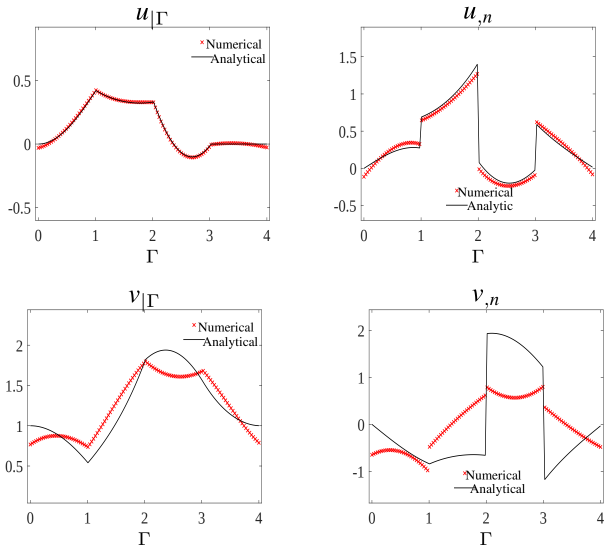
□ $\Omega_d = 15 \times 15$: data

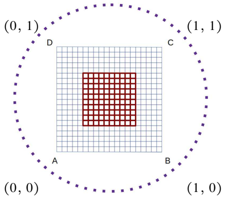
□ Ω_i : unknowns

● source points

$M = 8$ and $d = 5$

Reconstructions of the boundary conditions of a square domain by the MFS from interior noisy data measurements ($\delta = 3\%$)





$\Omega = 30 \times 30$

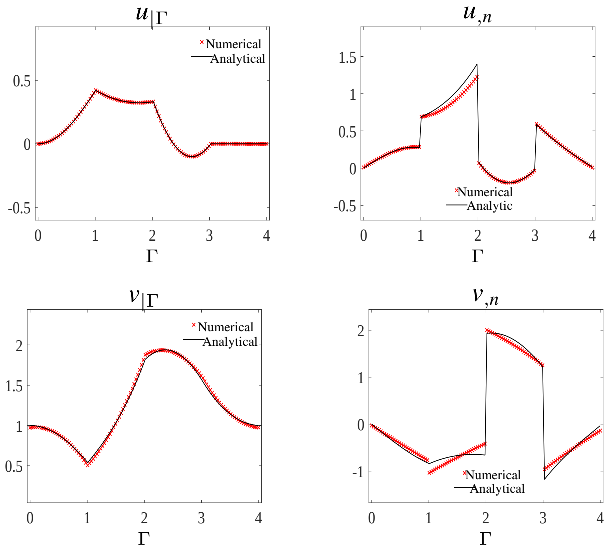
□ $\Omega_d = 15 \times 15$: data

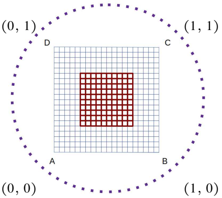
□ Ω_i : unknowns

● source points

$M = 8$ and $d = 5$

Reconstructions of the boundary conditions of a square domain by the MFS from interior exact data measurements





$\Omega = 30 \times 30$

$\Omega_d = 15 \times 15$: data

Ω_i : unknowns

● source points

$M = 8$ and $d = 5$

Theorem (Théorème de Holmgren – unicité du prolongement harmonique)

Soit $u \in H^2$ *une solution du problème* $P(u) = 0$ *où les coefficients de* P *sont analytiques et* $u = 0$ *sur une courbe* Γ *non-caractéristique de classe* C^1. *Alors* u *est identiquement nulle dans un voisinage de chaque point de* Γ.

Remark

Le théorème de Holmgren s'applique en particulier aux opérateurs elliptiques puisqu'ils n'admettent pas de courbes caractéristiques.