Department of Mathematics

On Some Spectral Sets Important to the Scattering Theory

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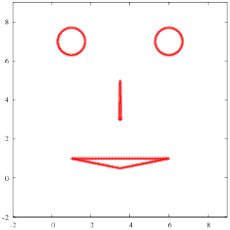
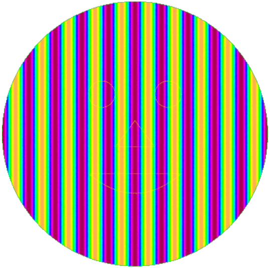
Scattering of Waves

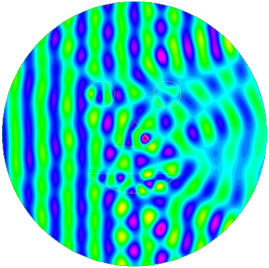
Courtesy of H. Haddar

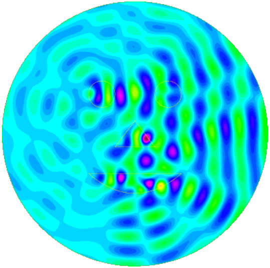


Scattering Media

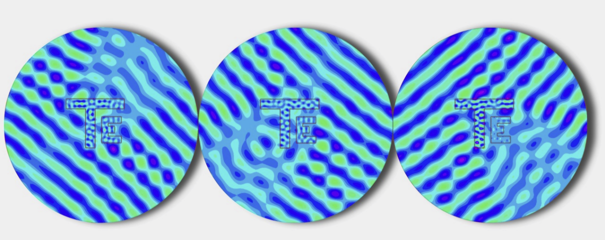


Probing (incident) Plane Field

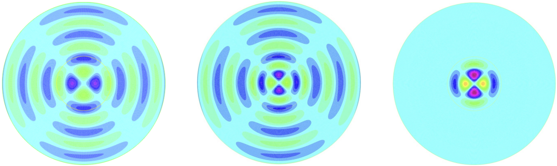




Inverse Scattering Problem



But it could happen



incident field　　total field　　scattered field

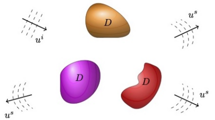
Literature

F. Cakoni, D. Colton and H. Haddar (2021) *Transmission Eigenvalues*, AMS Notices, October Issue.

F. Cakoni and M. Vogelius (2021), Singularities almost always scatter: Regularity results for non-scattering inhomogeneities, *Communications in Pure and Applied Math* (to appear).

F. Cakoni, D. Colton and H. Haddar (2020), A duality between scattering poles and transmission eigenvalues in scattering theory, *Proc. R. Soc. A*.

Scattering by an Inhomogeneous Media



$k := \omega/c_0 > 0$ is the wave number

(real) refractive index $\sqrt{n} := c_0/c(x)$

$n \in L^\infty(\mathbb{R}^3)$ s.th. $\mathrm{Supp}(n-1) = \overline{D}$

∂D is Lipschitz

- The incident field v satisfies the Helmholtz equation

$$\Delta u^i + k^2 u^i = 0 \qquad \text{in } \mathbb{R}^3 \setminus \mathcal{O}$$

$\mathcal{O}$ set of zero measure in the exterior of D: examples $u^i(x) = e^{ikx\cdot\hat{y}}$ or $u^i(x) = \Phi(x,y) := \frac{e^{ik|x-y|}}{|x-y|}$

- The total field $U = u + u^i$ satisfies

$$\Delta U + k^2 n(x) U = 0 \qquad \text{in } \mathbb{R}^3$$

- The scattered field u satisfies

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \quad \text{Sommerfeld radiation condition}$$

Scattering by an Inhomogeneous Media

For the given incident field u^i, the scattered field satisfies

$$\Delta u + k^2 n u = k^2(1-n)u^i \qquad \text{in } \mathbb{R}^3 \qquad + \text{SRC}$$

or $\Delta u + k^2 u = k^2(1-n)(u^i + u)$

Note Only $u^i|_D \in L^2(D)$ matters. Let $v := u^i|_D$ and in general $v \in \left\{L^2(D) \text{ distributional solution to } \Delta v + k^2 v = 0\right\}$

Lippmann-Schwinger equation

$$u = k^2 \mathbb{T}_{n,k}(u+v)$$

where

$$\left(\mathbb{T}_{n,k}\theta\right)(x) = \int_D (n(y)-1)\theta(y)\frac{e^{ik|x-y|}}{4\pi|x-y|}\,dy \qquad x \in \mathbb{R}^3$$

Scattering by an Inhomogeneous Media

Total field $u + v$ in D solves $\left[\mathbb{I} - k^2 \mathbb{T}_{n,k}\right](u + v) = v$

- $\mathbb{T}_{n,k} : L^2(D) \to L^2(D)$ is compact.
- Fredholm alternative applies to $\left[\mathbb{I} - k^2 \mathbb{T}_{n,k}\right]$, i.e. injectivity implies bounded invertibility
- The norm $\mathbb{T}_{n,k} : L^2(D) \to L^2(D)$ is bounded independently of k.
- By Rellich's Lemma $\left[\mathbb{I} - k^2 \mathbb{T}_{n,k}\right]$ is injective for $\Im(k) \geq 0$
- In particular for $\Im(k) \geq 0$

$$\left[\mathbb{I} - k^2 \mathbb{T}_{n,k}\right]^{-1} : v \mapsto (u + v)$$

$v \in \left\{L^2(D) \text{ distributional solution to } \Delta v + k^2 v = 0\right\}$

Scattering poles

- Recall $\left[\mathbb{I} - k^2\mathbb{T}_{n,k}\right] : \mathbb{C} \to \mathcal{L}(L^2(D))$.
- Analytic Fredhom Theory implies the kernel of $\left[\mathbb{I} - k^2\mathbb{T}_{n,k}\right]$ may be nontrivial for a possibly discrete set of $k \in \mathbb{C}$ with $\Im(k) < 0$ having no finite accumulation point.
- Therefore $\left[\mathbb{I} - k^2\mathbb{T}_{n,k}\right]^{-1}$ is meromorphic in $k \in \mathbb{C}$, and its poles in $\Im(k) < 0$ are the so-called scattering poles or resonances.

At a scattering pole $k \in \mathbb{C}$, $\Im(k) < 0$ there exists non-zero solution to the homogeneous scattering problem, or in other words there is

non-zero scattered field $u \neq 0$ with zero incident field $u^i = 0$

Scattering poles constitute a fundamental part of scattering theory.

S. Dyatlov - M. Zworski (2019), *Mathematical Theory of Scattering Resonances*, AMS.

Non-scattering k and Transmission Eigenvalues

If $k \in \mathbb{C}$ is such that the kernel of $\mathbb{I} - k^2 \mathbb{T}_{n,k}$ is trivial then

$$\left[\mathbb{I} - k^2 \mathbb{T}_{n,k}\right]^{-1} : v \mapsto (u + v) \qquad \text{in } L^2(D)$$

$v \in \left\{L^2(D) \text{ distributional solution to } \Delta v + k^2 v = 0\right\}$

Then u defined by

$$u(x) = \int_D (n(y) - 1) \left[(\mathbb{I} - k^2 \mathbb{T}_{n,k})^{-1} v\right](y) \frac{e^{ik|x-y|}}{4\pi|x-y|}\, dy \qquad x \in \mathbb{R}^3$$

is the unique $u \in H^2_{loc}(\mathbb{R}^3)$ (radiating for $\Im(k) \geq 0$) solution of

$$\Delta u + k^2 n u = k^2 (1 - n) v \qquad \text{in } \mathbb{R}^3$$

- k is a non-scattering wave number if $u = 0$ in $\mathbb{R}^3 \setminus D$ for $v := u^i|_D$
- k is a transmission eigenvalue if $u = 0$ in $\mathbb{R}^3 \setminus D$ for any v as above.

Non-scattering k and Transmission Eigenvalues

For the inhomogeneity (n, D), $u = 0$ in $\mathbb{R}^3 \setminus D$ implies that there is a compactly supported distributional solution u

$$\Delta u + k^2 n u = k^2(1 - n)v \qquad \text{in } \mathbb{R}^3$$

$$u \equiv 0 \qquad \text{in } \mathbb{R}^3 \setminus D.$$

corresponding to v such that

- if $\Delta v + k^2 v = 0$ in $\mathbb{R}^3$ (or open $\Omega \supset \overline{D}$) then k is a non-scattering wave number: at this wave number the incident field v is not scattered by this inhomogeneity.
- If $\Delta v + k^2 v = 0$ in D then k is a transmission eigenvalue.

Non-scattering wave numbers form a subset of of transmission eigenvalues.

Physically only real non-scattering wave numbers $k > 0$ are of relevance.

Spherical Geometry

$D := B_1(0)$, $n(r)$, incident field $v = j_\ell(k|x|)Y_\ell(\hat{x})$

$$u^s(x) := \frac{C(k;n,\ell)}{W(k;n,\ell)} h_\ell^{(1)}(k|x|)Y_\ell(\hat{x}) \qquad |x| \geq 1$$

$$C(k;n,\ell) = \text{Det}\begin{pmatrix} y_\ell(1;k,n) & j_\ell(k) \\ y_\ell'(1;k,n) & kj_\ell'(k) \end{pmatrix}$$

$$W(k;n,\ell) = \text{Det}\begin{pmatrix} y_\ell(1;k,n) & h_\ell^{(1)}(k) \\ y_\ell'(1;k,n) & kh_\ell^{(1)'}(k) \end{pmatrix}$$

with $y_\ell(r;k,n)$ the solution (regular at $r=0$) of

$$y'' + \frac{2}{r} + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y = 0.$$

- k for which $C(k;n,\ell) = 0$ are non-scattering wave numbers with $v = j_\ell(k|x|)Y_\ell(\hat{x})$ as incident wave.
- k for which $W(k;n,\ell) = 0$ are scattering poles.

Transmission Eigenvalues for Spherical Geometry

If k is such that $C(k; n, \ell) = 0$, i.e. a non-scattering wave number then $v = j_\ell(k|x|)Y_\ell(\hat{x})$ and $w = y_\ell(|x|; k, n)Y_\ell(\hat{x})$ solve

The Transmission Eigenvalue Problem

$$\Delta v + k^2 v = 0 \quad \text{and} \quad \Delta w + k^2 n w = 0 \qquad |x| < 1$$

$$v = w \quad \text{and} \quad \frac{\partial v}{\partial |x|} = \frac{\partial w}{\partial |x|} \qquad |x| = 1$$

For spherically symmetric media the set of non-scattering wave numbers and the set of transmission eigenvalues coincide

There exists infinitely many real and infinitely many complex zeros of $C(k; n, \ell) = 0$ with ∞ as the only accumulation point COLTON-LEUNG (2017). Thus the transmission eigenvalue problem is non-selfadjoint.

Non-scattering results for spherical media

- If $n(r) \in C^2[0, 1]$ for each incident wave of the form $v = j_\ell(k|x|)Y_\ell(\hat{x})$ there exists infinitely many $k > 0$ accumulating only at $+\infty$ for which $B_1(0), n(r)$ renders this v non-scattering.
- Non-scattering incident waves for a ball are Herglotz wave function (superposition of plane waves)

$$v(x) := \int_{\mathbb{S}} g(\hat{y}) e^{ikx\cdot\hat{y}}, \qquad \hat{y} = y/|y|$$

D. Colton and R. Kress (2019), *Inverse Acoustic and Electromagnetic Scattering Theory*, 4th Edition, Springer.

- In $\mathbb{R}^2$ and $n \neq 1$ constant, it is shown that if the disk is perturbed to an ellipse with arbitrary small eccentricity, there are at most finitely many $k > 0$ for which a Herglotz wave function with a fixed, smooth density can be non-scattering.

M. Vogelius and J. Xiao (2021), *Finiteness results concerning non-scattering wave numbers for incident plane and Herglotz waves*, SIAM J. Math Analysis.

General Media

Non-scattering Configuration

Find $u \in H_0^2(D)$ (H^2 with compact support $\overline{D}$)

$$\Delta u + k^2 n u = k^2(1-n)v \quad \text{in } D$$

with a nonzero v physical incident wave which satisfies

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \mathcal{O}$$

for Lipschitz boundary $u \in H_0^2(D)$ means $u = 0$, $\dfrac{\partial u}{\partial \nu} = 0$ on ∂D

If k is non-scattering wave number then u and $v|_D$ satisfy

Transmission Eigenvalue Problem

There are nonzero $u \in H_0^2(D)$ and $v \in L^2(D)$ such that

$$\Delta u + k^2 n u = k^2(1-n)v \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D$$

The Transmission Eigenvalue Problem

Set $U = u + v$ then nonzero $v \in L^2(D)$ and $U \in L^2(D)$ satisfy

$$\begin{aligned} \Delta v + k^2 v = 0 & \quad \text{in} \quad D \\ \Delta U + k^2 n U = 0 & \quad \text{in} \quad D \\ U = v & \quad \text{on} \quad \partial D \\ \frac{\partial U}{\partial \nu} = \frac{\partial v}{\partial \nu} & \quad \text{on} \quad \partial D \end{aligned}$$

TE can be also viewed as values of $k \in \mathbb{C}$ for which the operator

$$\mathcal{N}_{k,n} - \mathcal{N}_{k,1} \qquad \text{has non-trivial kernel}$$

where $\mathcal{N}_{\tau,q} : f \mapsto \dfrac{\partial \varphi}{\partial \nu}$ (the Dirichlet-to-Neuman operator) with φ

$$\Delta \varphi + k^2 q \varphi = 0 \quad \text{in } D \qquad \varphi = f \quad \text{on } \partial D$$

The State of the Art of TE problem

The transmission eigenvalue problem is non-selfadjoint.

Main Assumption $n - 1$ is one sign in a neighborhood of ∂D

- If $n \in L^\infty(D)$, and ∂D Lipschitz, transmission eigenvalues are discrete with ∞ as the only possible accumulation point.

  SYLVESTER (2012), KIRSCH (2014)

- If $n \in C^1$ near ∂D smooth, completeness of generalized eigenfunctions and Weyl's law for counting function are known.

  ROBIANNO (2013), H.M. NGUYEN-J. FORNEROD (2022)

- If $n \in C^\infty(\overline{D})$ and ∂D is C^∞, transmission eigenvalues lie in a strip around the real axis.

  VODEV (2018)

F. CAKONI, D. COLTON AND H. HADDAR (2016), *Inverse Scattering Theory and Transmission Eigenvalues*, CBMS-NSF, SIAM Publication.

Existence of Real Transmission Eigenvalues

Important to our discussion is whether real transmission eigenvalues exist.

Theorem (Cakoni-Gintides-Haddar)

Assume $n \in L^\infty(D)$, ∂D Lipschitz, and $n-1$ is one sign uniformly in D.

- There exists an infinite sequence of real transmission eigenvalues $\{k_j\}_{j\in\mathbb{N}}$ accumulating at $+\infty$.
- If $n>1$ than $k_1 > \dfrac{\lambda_1(D)}{sup_D n}$, and if $0<n<1$ than $k_1 > \lambda_1(D)$

$\lambda_1(D)$ is the smallest Dirichlet eigenvalue of $-\Delta$ in D

In this case, eliminating v we get: find $u \in H^2(D)$ such that

$$(\Delta + k^2)\frac{1}{n-1}(\Delta + k^2 n)u = 0 \quad \text{in } D$$

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D$$

F. Cakoni, D. Gintides and H. Haddar (2010), *The existence of infinite discrete set of transmission eigenvalues*, SIAJ. Math Anal.

TE and Non-Scattering

$\sup(1 - n) = \overline{D}$

$k > 0$ is a transmission eigenvalue

if there are nonzero v and $u \in H_0^2(D)$ such that

$$\begin{aligned} \Delta v + k^2 v &= 0 && \text{in } D \\ \Delta u + k^2 n u &= k^2(1 - n)v && \text{in } D \\ u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial D \end{aligned}$$

$k > 0$ is a non-scattering wave number

if there are nonzero v and $u \in H_0^2(D)$ such that

$$\begin{aligned} \Delta v + k^2 v &= 0 && \text{in } \Omega \supset \overline{D} \\ \Delta u + k^2 n u &= k^2(1 - n)v && \text{in } D \\ u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial D \end{aligned}$$

Non-existence of Non-scattering Wave Numbers

Negative result in this context means that non-scattering configuration does NOT exist

First negative result obtained for corners Blåsten-Päivärinta-Sylvester (2013)

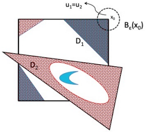
If D contains a boundary point $x_0 \in \partial D$ that is a corner in $\mathbb{R}^2$, or a vertex, conical corner, edge point in $\mathbb{R}^3$, and $n(x_0) \neq 1$ and $n \in C^{1,\alpha}$ locally in $B_\epsilon(x_0)$, then every incident wave is scattered by D, n.

No assumption on the incident field v is needed!

This negative result implies that scattering data due to one single incident plane wave uniquely determines the support of convex polyhedron inhomogeneities.

Hu-Salo-Vesalainen (2016), Elschner-Hu (2017), (2018)

Blåsten (2018), Cakoni-Xiao (2019)



Two Techniques for Corner Scattering

- Based on CGO (rapidly decaying) solutions of the Helmholtz equation.

  BLÅSTEN-PÄIVÄRINTA-SYLVESTER (2013), PÄIVÄRINTA-SYLVESTER-VESALAINEN (2017), BLÅSTEN (2018), CAKONI-XIAO (2019), XIAO (2021)

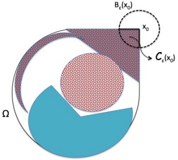
  CGO solution is used as test function w in

$$\int_{\mathcal{C}_\epsilon} (1-n) v \varphi \, dx = \int_{\mathcal{K}_\epsilon} \varphi \frac{\partial u}{\partial \nu} - u \frac{\partial \varphi}{\partial \nu} \, ds$$

  to control the boundary terms, where u and v are transmission eigenfunctions.

- Based on singularity analysis of the transmission eigenfunctions in a neighborhood of the boundary singularity.

  ELSCHNER-HU (2017), (2018)



In both methods a contradiction is achieved if v is assumed to solve the Helmholtz equation in $B_\epsilon(x_0)$.

Singularities Scatter

For general domains D this question is only recently studied.

- In $\mathbb{R}^2$ and constant $n \neq 1$, $C^{2,\alpha}$ strictly convex inhomogeneities scatter plane waves, except for at most finitely many wave numbers.

  M. Vogelius and J. Xiao (2021), Finiteness results concerning non-scattering wave numbers for incident plane and Herglotz waves, *SIAM J. Math Analysis*.

- If ∂D contains a smooth portion of high curvature, (D, n) scatter all waves with sufficiently large modulus near this portion.

  E. Blåsten and H. Liu (2021),Scattering by curvatures, radiationless sources, transmission eigenfunctions and inverse scattering problems, *SIAM Math Analysis*.

- Using free boundary methods

  F. Cakoni and . Vogelius (2021), Singularities almost always scatter: Regularity results for non-scattering inhomogeneities, *Communications in Pure and Applied Math* (to appear).

  M. Salo and H. Shahgholian (2021), Free boundary methods and non-scattering phenomena, *Research in the Mathematical Sciences*.

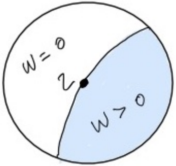
Free Boundary Methods

Free boundary methods apply to problems

$$\Delta w = f\chi_{\{w\neq 0\}} \qquad \text{in } B_r(z)$$

$$z \in \partial\{w = |\nabla w| = 0\}$$

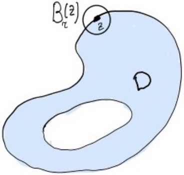
where f is Lipshitz and $w \geq 0$.



For our scattering problem, $z \in \partial D$ reachable from the unbounded component of $\mathbb{R}^d \setminus D$

$$f := -k^2 w + k^2(1 - n)\Re(v)$$

with $w := \Re(u)$ real part of the scattered field



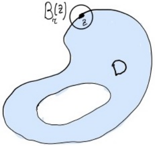
Almost All Singularities Scatter

Let ∂D be Lipschitz, $n \in L^\infty(D)$. The incident field v is scattered if

$$\Delta w + k^2 n w = k^2(1-n)\Re(v) \quad \text{in } D \cap B_r(z)$$

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial D \cap B_r(z)$$

for some $z \in \partial D$ with $(1-n(z))\Re(v(z)) \neq 0$



does NOT hold

incident field v is real analytic as solution of $\Delta v + k^2 v = 0$ in $\Omega \supset \overline{D}$

(Cakoni-Vogelius 2021)

The incident field v is scattered if there is $z \in \partial D$ where $n(z) \neq 1$ and $v(z) \neq 0$ such that

1. n and v are real analytic around z and $\partial D \cap B_r(z)$ is not analytic for any $r > 0$.
2. n and v in $C^{m+\mu}(\overline{D} \cap B_r(z))$ for $m \geq 1$, $0 < \mu < 1$ and $\partial D \cup B_r(z)$ is not in $C^{m+1+\mu}$ for any $r > 0$.

Remarks

Remark 1: The above results can be interpreted as statement of regularity up to the boundary of the transmission eigenfunction $v \in L^2(D)$ of

$$\Delta u + nu = k^2(1-n)v \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D.$$

For example, if n is real analytic in a neighborhood of $z \in \partial D$ and $\partial D \cap B_r(z)$ is not analytic for any $r > 0$, then v can not be real analytic in any neighborhood of z, unless $v(z) = 0$.

Remark 2: The non-vanishing condition $v(z) \neq 0$ on incident waves which is essential to free boundary methods is restrictive.

It is satisfied:

- for incident plane waves $v(x) = e^{ikx\cdot\hat{y}}$
- for some superposition of plane waves (Herglotz functions) $v(x) := \int_{\mathbb{S}} g(\hat{y})e^{ikx\cdot\hat{y}}d\hat{y}$. k is not a Dirichlet eigenvalue for $-\Delta$ in D

  Open Problem: For given n, D what is the intersection of the sets of Dirichlet and real transmission eigenvalues?

Ideas of Proof

Higher regularity is provided by the celebrated paper by Kinderlehrer and Niremberg (1977) which roughly state that if $\partial D \cap B_r(0)$ is C^1 and u is C^2 solution of the elliptic problem $z \to 0 \in \partial D$

$$F(x, u, Du, D^2u) = 0, \quad F(0, 0, \cdots) \neq 0 \quad \text{in} \quad D \cap B_r(0) \quad F \text{ is } C^1\text{-real valued}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial D \cap B_r(0)$$

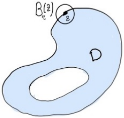
the boundary must one Hölder order higher regular than F.

To achieve starting regularity we use Caffarelli's result for $F : \Delta u = f$

Theorem (Caffarelli - 1977)

Let $\partial D \cap B_r(0)$ be Lipschitz, f has Lipschitz extension f^* in a neighborhood of $\overline{D} \cap B_r(0)$, s.th $f^* \geq \alpha > 0$, and u is $C^{1,1}$ and $u \geq 0$.

Then for some $r' < r$, $\partial D \cap B_{r'}(0)$ is C^1 and $u \in C^2(\overline{D} \cap B_{r'}(0))$.



Ideas of Proof

For our problem $f := -k^2 n w + k^2 (1-n) \Re(v)$, where w is the real part of the scattered field and v is the incident field.

- We first prove that $w \in C^{1,1}$.
- Then $f \in C^{1,1}(\overline{D} \cap B_r(z))$ with non-degeneracy condition $[(1-n)\Re(v)](z) \neq 0$ implies one sign condition on w

Our approach is based on

S. Williams (1981), *Analyticity of the boundary for Lipschitz domains without the Pompeiu property*, Indiana Univ. Math. J.

More optimal starting regularity assumptions on ∂D are used in

M. Salo and H. Shahgholian (2021), Free boundary methods and non-scattering phenomena, Res. Math. Sci.

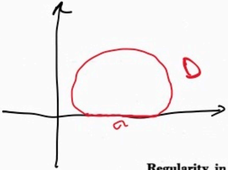
where based on the results in

J. Andersson, E. Lindgren and H. Shahgholian (2021), Optimal regularity for the non-sign obstacle problem, CMAP (2012)

it is proven that assuming $\overline{D} \cap B_r(z)$ is solid domain, then either free boundary $\overline{D} \cap B_r(z)$ is smooth or is thin at z.

Historical Connections

1959 H. Lewy



$$\begin{cases} \Delta u + m(x,y)\,u = 0 \\ \Delta v = 0 \end{cases} \quad \text{in } D$$

$$\begin{cases} u(x,0) = v(x,0) \\ u_y(x,0) = v_y(x,0) \end{cases} \quad \text{on } \partial$$

Regularity in Free Boundary Problems.

D. KINDERLEHRER (*) - L. NIRENBERG (**) (1)

dedicated to Hans Lewy

1. – Introduction.

This paper is concerned with the local regularity of free boundary hypersurfaces, in n dimensional space, for elliptic and parabolic second order partial differential equations. In a free boundary problem, part of the problem is to determine the position and regularity of the free boundary. In order to do this one is usually provided with more boundary conditions at the free boundary than one has for a known boundary. We begin with some simple model examples. In all but Ex. 4, Ω represents a domain in R^n, $n>2$, with the origin on its boundary $\partial\Omega$. All our discussion is purely local, near the origin, and our results have the following nature: assuming *some* degree of regularity of the free boundary and of the solution near it, specifically conditions (I) and (II) below, we prove further regularity. The conditions (I, II) are rather strong and not always satisfied in practice as we indicate. In a neighbourhood of the origin we assume:

Connection to Schiffer's Conjecture

Non-scattering: Given v satisfying $\Delta v + k^2 v = 0$ in $\mathbb{R}^d$, the problem

$$\begin{aligned} \Delta u + k^2 n u = k^2(1-n)v \quad & \text{in } D \\ u = 0, \ \frac{\partial u}{\partial \nu} = 0 \quad & \text{on } \partial D \end{aligned}$$

has no solution for any $k > 0$.

D has Schiffer's property if the problem

$$\begin{aligned} \Delta w + \lambda w = 1 \quad & \text{in } D \\ w = 0, \ \frac{\partial w}{\partial \nu} = 0 \quad & \text{on } \partial D \end{aligned}$$

has no solution for any λ.

Conjecture: The only simply connected domain in $\mathbb{R}^d$ that fails to have Schiffer's property are balls. Williams (1981), Berenstein-Yang (1987), Vogelius (1994)

Schiffer's property is related to Pompeiu property (integral geometry problem)

Remarks on the Near Field Scattering Operator

Consider single layer potential incident waves (superposition of point sources $\Phi(\cdot, y)$)

$$v_g(x) := \int_{\partial\Omega} g(y)\Phi(x, y)\, ds_y$$

where $\partial\Omega$ is the boundary of an open region $\Omega \supset \overline{D}$.

with the corresponding scattered field u_g solving

$$\Delta u_g + k^2 n u_g = k^2(1 - n)v_g \qquad \text{in } \mathbb{R}^3, \qquad \text{with SRC}$$

The near field incoming-to-outgoing scattering operator is defined by

$$\mathcal{N}_k : g \in L^2(\partial\Omega) \mapsto u_g \in L^2(\partial\Omega)$$

Remarks on the Near Field Scattering Operator

Theorem

The near field scattering operator $\mathcal{N}_k : L^2(\partial\Omega) \to L^2(\partial\Omega)$ corresponding to a given inhomogeneity D, n at a wave number k is

not injective and does not have dense range

if and only if there exist a nontrivial single layer potential incident wave

$$v_g(x) := \int_{\partial\Omega} g(y)\Phi(x,y)\,ds_y$$

that is not scattered by D, n.

Relative Scattering Operator and Special Sets of k

- Transmission Eigenvalues: related to the kernel

$$v \in \{L^2(D) : \Delta v + k^2 v = 0\} \mapsto u|_{\partial\Omega} \qquad \text{where } u \text{ solves}$$

$$\Delta u + k^2 n u = k^2(1-n)v \quad \text{in } \mathbb{R}^3 \quad \text{SRC}$$

- Non-scattering Frequencies: related to the kernel

$$\mathcal{N}_k : g \mapsto u_g|_{\partial\Omega}$$

- Scattering Poles: they are the poles of the meromorphic valued operator $k \to \mathcal{N}_k \in \mathcal{L}(L^2(\partial\Omega))$

It is possible to characterize the scattering poles as 'non-scattering frequencies" for an appropriate interior scattering problem and outgoing incident field.

F. Cakoni, D. Colton and H. Haddar (2020), A duality between scattering poles and transmission eigenvalues in scattering theory, *Proc. R. Soc. A*.

Dual Characterization of the Scattering Poles

Given n and D, k is a scattering pole if and only if the homogeneous problem

$$\Delta w + k^2 n w = 0 \qquad \text{in } \mathbb{R}^3$$

$$w = \mathcal{SL}^k_{\partial D}(\partial w/\partial \nu) - \mathcal{DL}^k_{\partial D}(w) \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}$$

has non-trivial solution $w \in H^2_{loc}(\mathbb{R}^3)$

For any $k \in \mathbb{C}$ consider incident fields $w \in H^2_{loc}(\mathbb{R}^3 \setminus \overline{D})$ that satisfies

$$\Delta w + k^2 w = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}$$

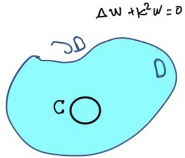
$$w = \mathcal{SL}^k_{\partial D}(\partial w/\partial \nu) - \mathcal{DL}^k_{\partial D}(w) \text{ in } \mathbb{R}^3 \setminus \overline{D}$$

Such incident field are e.g. $w := \Phi(\cdot, z)$ the fundamental solution of the Helmholtz equation, or superposition $w := \int_{\mathcal{C}} \Phi(\cdot, z)\varphi(z)dz$ for an analytic curve $\mathcal{C} \subset D$.

$\mathcal{SL}^k_{\partial D}$ and $\mathcal{DL}^k_{\partial D}$ are the single and double layer boundary potentials.

A dual framework

Then, assuming k is not a transmission eigenvalue, consider the interior scattering problem: find "total field" $u \in L^2(D)$ and "scattered field" $v \in L^2(D)$ with $u - v \in H^2(D)$ such that



$$\Delta u + k^2 n u = 0 \;\text{ and }\; \Delta v + k^2 v = 0 \qquad \text{in } D$$

$$u - v = w \;\text{ and }\; \frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \qquad \text{on } \partial D$$

(Dual Framework)

- $k \in \mathbb{C}$ is a scattering pole if and only if the kernel of the mapping

$$w \mapsto v_w|_{\mathcal{C}}$$

  is non-trivial (dual to transmission eigenvalue characterization).
- Mapping $\varphi \mapsto v_\varphi$ where v_φ satisfies the above scattering problem with $w := \int_{\mathcal{C}} \Phi(\cdot, z)\varphi(z)dz$ replaces the near field operator.

Open Problem: What is the intersection of the set of scattering poles with the set of complex transmission eigenvalues.