Recovery of remanent magnetizations using geometric measure theory, l1 and total variation regularization

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Overview

The main problem we are interested in is the one of recovering the remanent magnetization $\mathbf{M}$ of a rock sample from measurements of its magnetic field $\mathbf{B}$.

Using the magnetic potential Φ which satisfies

$$\mathbf{B} = \mu_0 (\mathbf{M} - \operatorname{grad} \Phi)$$
$$\Delta \Phi = \operatorname{div} \mathbf{M}$$

we study the inverse problem for the recovery of $\mathbf{M}$ when this magnetization is modeled by a vector valued measure.

We use methods for recovering $\mathbf{M}$ based on total variation regularization.

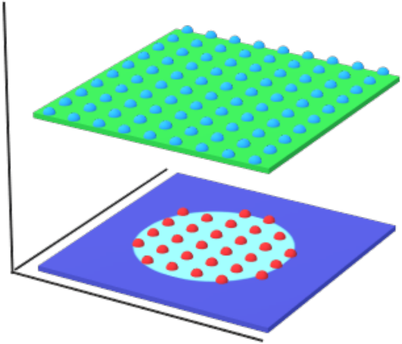
Research Groups

This work is part a collaborative project from the following:

- Eduardo Andrade Lima and Benjamin Weiss of the MIT Department of Earth, Atmospheric, and Planetary Sciences.
- Laurent Baratchart, Sylvain Chevillard and Juliette Leblond of the FACTAS, (formerly APICS) team in INRIA Sophia Antipolis.
- Doug Hardin and Edward Saff of Vanderbilt University

The measurements are usually obtained form a Scanning Magnetic Microscope (SMM) such as the instrument used by the research team in the MIT.

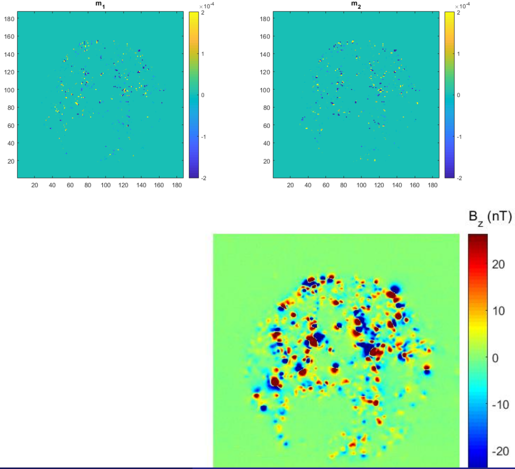
SMM setup



Q will denote the set on which we take the measurement.

S will denote a super set of the magnetization support.

Example of a reconstruction



Theoretical framework

The *magnetic moment*
is a vector $\mathbf{m}$ that measures the strength and direction of objects that generate magnetic fields.

A *magnetic dipole*
can be viewed as the limit of a current loop that shrinks to a point with constant moment.

For a dipole at the point $\mathbf{y} \in \mathbb{R}^3$ with magnetic moment $\mathbf{m}$, the field at the point $\mathbf{x}$ is

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\left(\frac{\mathbf{m}}{|\mathbf{x}-\mathbf{y}|^3} - 3(\mathbf{x}-\mathbf{y})\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{m}}{|\mathbf{x}-\mathbf{y}|^5}\right).$$

Magnetizations

The *magnetization* of an object is a density for the magnetic moment.

We will model magnetizations by vector valued Borel measures.

$\mathcal{M}(S)$ denotes the space of Borel measures supported on $S \subset \mathbb{R}^3$.

Total variation of Magnetizations

Each $\mathbf{M} \in \mathcal{M}(S)^3$ is of the form $\mathrm{d}\mathbf{M} = \mathbf{u_M}\mathrm{d}|\mathbf{M}|$, where $|\mathbf{M}|$ is a positive Borel measure supported on S and $|\mathbf{u_M}| = 1$ $|\mathbf{M}|$-a.e.

In this case the *Total Variation Norm* of a measure $\mathbf{M}$ is

$$\|\mathbf{M}\|_{TV} := |\mathbf{M}|(\mathbb{R}^3).$$

Note that in the case where $\mathbf{M} = \operatorname{grad} u$ then $\|\mathbf{M}\|_{TV}$ coincides with the total variation of u.

Definition Φ and $\mathbf{B}$ as distributions on $\mathbb{R}^3$

For a magnetization $\mathbf{M}$ we will define $\Phi(\mathbf{M}) \in L^1_{loc}(\mathbb{R}^3)$, the **scalar magnetic potential of M** as the unique distribution that satisfies

$$\Delta\Phi = \operatorname{div}\mathbf{M}$$

and for any point $\mathbf{x}$ not in the support of $\mathbf{M}$

$$\Phi(\mathbf{M})(\mathbf{x}) = \int (\operatorname{grad}\Gamma)(\mathbf{x}-\mathbf{y}) \cdot \mathrm{d}\mathbf{M}(\mathbf{y}) = \frac{1}{4\pi}\int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \cdot \mathrm{d}\mathbf{M}(\mathbf{y}),$$

where $\Gamma(\mathbf{x}) := -1/(4\pi|\mathbf{x}|)$ is the Newtonian kernel. Then we define the magnetic field $\mathbf{B}(\mathbf{M})$ generated by $\mathbf{M}$ as

$$\mathbf{B}(\mathbf{M}) := \mu_0\left(\mathbf{M} - \operatorname{grad}\Phi(\mathbf{M})\right).$$

Definition Forward operator

Consider a finite, positive Borel measure ρ with support contained in Q and let $\mathbb{A} : \mathcal{M}(S)^3 \to L^2(Q, \rho)$ be the operator defined by

$$\mathbb{A}(\mathbf{M})(\mathbf{x}) := \mathbf{v} \cdot \mathbf{B}(\mathbf{M})(\mathbf{x}), \qquad \mathbf{x} \in Q. \tag{1}$$

Since $\mathbf{K}_{\mathbf{v}}$ is continuous on $\mathbb{R}^3 \setminus \{0\}$ and Q and S are positively separated, it follows that $B_{\mathbf{v}}$ is continuous on Q and consequently $\mathbb{A}$ does indeed map $\mathcal{M}(S)^3$ into $L^2(Q, \rho)$.

S-silent

We say that magnetizations $\mathbf{M}, \mathbf{N} \in \mathcal{M}(S)^3$ are **S-equivalent** if $\mathbf{B}(\mathbf{M})$ and $\mathbf{B}(\mathbf{N})$ agree on $\mathbb{R}^3 \setminus S$ in which case we write

$$\mathbf{M} \stackrel{S}{\equiv} \mathbf{N}.$$

A magnetization $\mathbf{M}$ is said to be **S-silent** (or **silent in $\mathbb{R}^3 \setminus S$**) if it is S-equivalent to the zero magnetization; i.e., if $\mathbf{B}(\mathbf{M})$ vanishes on $\mathbb{R}^3 \setminus S$.

Regularization problems

To fix notation, for $\mathbf{M} \in \mathcal{M}(S)^3$, let

$$\mathfrak{M}(\mathbf{M}) := \inf\{\|\mathbf{N}\|_{TV} : \mathbf{N} \stackrel{S}{\equiv} \mathbf{M}\}. \tag{2}$$

Extremal Problem (EP-1)

Given $\mathbf{M}_0 \in \mathcal{M}(S)^3$, *find* $\mathbf{M} \stackrel{S}{\equiv} \mathbf{M}_0$ *such that* $\|\mathbf{M}\|_{TV} = \mathfrak{M}(\mathbf{M}_0)$.

Regularization problems

The second extremal problem involves minimizing the following functional defined for $\mathbf{M} \in \mathcal{M}(S)^3$, $f \in L^2(Q)$, and $\lambda > 0$, by

$$\mathcal{F}_{f,\lambda}(\mathbf{M}) := \|f - \mathbf{A}\mathbf{M}\|^2_{L^2(Q)} + \lambda \|\mathbf{M}\|_{TV}. \tag{3}$$

Extremal Problem (EP-2)

Given $f \in L^2(Q)$, find $\mathbf{M}_\lambda \in \mathcal{M}(S)^3$ such that

$$\mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda) = \inf_{\mathbf{M} \in \mathcal{M}(S)^3} \mathcal{F}_{f,\lambda}(\mathbf{M}). \tag{4}$$

Slender set

We will call a closed set $S \subset \mathbb{R}^3$ a **slender set** if $\mathcal{L}_3(S) = 0$ and each connected component C of $\mathbb{R}^3 \setminus S$ satisfies $\mathcal{L}_3(C) = \infty$.

Lemma

Let $S \in \mathbb{R}^3$ be closed and slender.
Then $\mathbf{M}$ is S-silent if and only if $\operatorname{div} \mathbf{M} = 0$.

S-silent not slender

Let S be the closed unit Euclidean ball centered at the origin, $\mathrm{M} = \mathcal{L}_3 \lfloor S$, $\mathbf{M} = (4\pi/3)^{-1}\mathrm{M}\mathbf{e}_1$ and $\mathbf{N} := \delta_0 \mathbf{e}_1$ where δ_0 is the Dirac delta at zero.

Using the mean value theorem it follows that both have magnetic potential

$$\frac{1}{4\pi}\frac{x_1}{|\mathbf{x}|^3}$$

so $\mathbf{M} - \mathbf{N}$ is S-silent.

However, $\mathbf{M} - \mathbf{N}$ is not divergence free.

Sparse in the sense of being purely 1-unrectifiable

A set $B \subset \mathbb{R}^n$ is **purely 1-unrectifiable** if $\mathcal{H}_1(f(\mathbb{R}) \cap B) = 0$ for every Lipschitz map $f : \mathbb{R} \mapsto \mathbb{R}^n$.

Theorem

Suppose $S \subset \mathbb{R}^3$ is a closed, slender set. If $\mathbf{M} \in \mathcal{M}(S)^3$ has support that is purely 1-unrectifiable and $\mathbf{N} \in \mathcal{M}(S)^3$ is S-equivalent to $\mathbf{M}$, then $\|\mathbf{N}\|_{TV} > \|\mathbf{M}\|_{TV}$ unless $\mathbf{N} = \mathbf{M}$.

Unidirectional

We say that $\mathbf{M}$ is **uni-directional** if $\mathbf{u_M}$ is constant a.e. with respect to $|\mathbf{M}|$.

Theorem

Let $S = \bigcup_{i=1}^{n} S_i$ for some disjoint closed sets $S_1, S_2, \ldots, S_n$ in $\mathbb{R}^3$ and suppose that S is either compact or slender. Let $\mathbf{M} \in \mathcal{M}(S)^3$ be such that $\mathbf{M}_i := \mathbf{M}\lfloor S_i$ is uni-directional for $i = 1, 2, \ldots, n$.
If $\mathbf{N} \in \mathcal{M}(S)^3$ is S-equivalent to $\mathbf{M}$, then $\mathbf{N}_i := \mathbf{N}\lfloor S_i$ and $\mathbf{M}_i$ are S_i-equivalent for $i = 1, 2, \ldots, n$, moreover

$$\|\mathbf{M}\|_{TV} \leq \|\mathbf{N}\|_{TV}, \tag{5}$$

with equality in (5) if and only if $\mathbf{N}_i$ is uni-directional in the same direction as $\mathbf{M}_i$ for $i = 1, 2, \ldots, n$. Furthermore, if S is slender and equality holds in (5), then $\mathbf{M} = \mathbf{N}$.

Oriented rectifiable curves

We call a Lipschitz mapping $\mathbf{c} : [0, \ell] \to \mathbb{R}^3$ a **rectifiable curve** and let $\mathbf{C} := \mathbf{c}([0, \ell])$ denote its image.

If $\mathbf{c}$ is an arclength parametrization of $\mathbf{C}$; i.e., if $\mathbf{c}$ satisfies

$$\mathcal{H}_1(\mathbf{c}([\alpha, \beta])) = \beta - \alpha, \qquad \forall [\alpha, \beta] \subset [0, \ell],$$

then we call $\mathbf{c}$ an **oriented rectifiable curve**.
We will call $\mathbf{R} \in \mathcal{M}(S)^3$ defined through the relation

$$\langle \mathbf{R}, \mathbf{f} \rangle = \int_0^\ell \mathbf{f}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \mathrm{d}t,$$

a **curve magnetization**.

Loop silent magnetizations

An oriented rectifiable curve will be called a **rectifiable Jordan curve** if it is injective on $[0, l)$ and such that $\mathbf{c}(0) = \mathbf{c}(l)$.

And its respective curve magnetization $\mathbf{M}$ is said to be a **loop silent magnetization**.

Any such magnetization is divergence free by the Gauss Green Theorem in the measure theoretical case.

Decomposition of divergence free magnetizations

Let $\mathcal{C} \subset \mathcal{M}(S)^3$ denote the collection of oriented rectifiable curves with topology inherited from $\mathcal{M}(S)^3$.
Suppose $\operatorname{div} \mathbf{M} = 0$ (as a distribution).

Smirnov [1, Theorem A] shows that $\mathbf{M}$ can be decomposed into elements from $\mathcal{C}$.

In particular, it can be proven that there is a positive Borel measure ρ on $\mathcal{C}$ such that

$$\mathbf{M}(B) = \int \mathbf{R}(B) \, \mathrm{d}\rho(\mathbf{R}),$$

Planar case

We will only consider planar magnetizations, that is magnetizations supported on a closed $S \subset \mathbb{R}^2 \times \{0\}$.

Recall that any such S is a slender set.

Decomposition of silent sources

Theorem

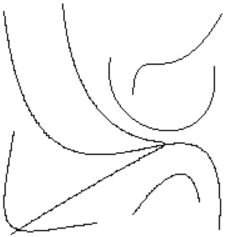
Let S be a closed subset of $\mathbb{R}^2$ and $\mathbf{M} \in \mathcal{M}(S)^3$ a silent magnetization. Then, for a.e. $t \in \mathbb{R}$ there is a countable family $\{\mathbf{N}_m^t\}$ of silent loop magnetizations such that

$$\|\mathbf{M}\|_{TV} = \int_{\mathbb{R}} \left(\sum \|\mathbf{N}_m^t\|_{TV} \right) \mathrm{d}t < +\infty, \tag{6}$$

and for any Borel set $B \subset \mathbb{R}^2$,

$$\mathbf{M}(B) = \int_{\mathbb{R}} \sum \mathbf{N}_m^t(B) \, \mathrm{d}t. \tag{7}$$

Tree-like set



A set is said to be **tree-like** if it contains no rectifiable Jordan curves.

Corollary

Let S be a closed subset of $\mathbb{R}^2$ and $\mathbf{M} \in \mathcal{M}(S)^3$ be S-silent. Then, the support of $\mathbf{M}$ contains a rectifiable Jordan curve. Hence, if S is tree-like the only S-silent magnetization is the zero magnetization.

Regularized inverse problem

l1 regularization

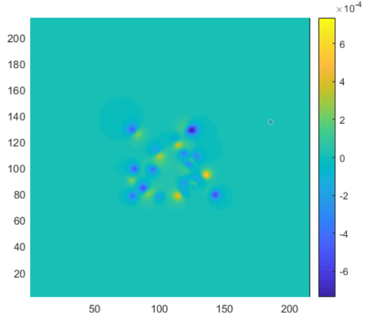
The problem we refer to EP2 has the following discreet analog:

$$\arg\min_{\mathbf{m}} \|b - A\mathbf{m}\|_2^2 + \lambda \sum_i \|\mathbf{m}_i\|_2.$$

$\|b - A\mathbf{m}\|_2^2$ measures the distance between the original field and the one generated by the reconstruction.

$\sum_i \|\mathbf{m}_i\|_2$ is used to enforce sparsity.

Illustration for purely 1-unrectifiable recovery



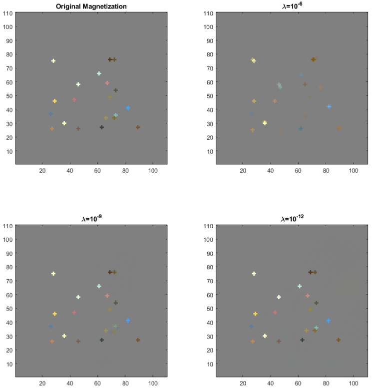
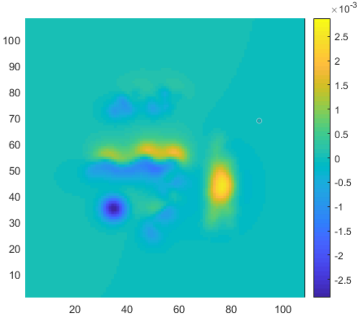
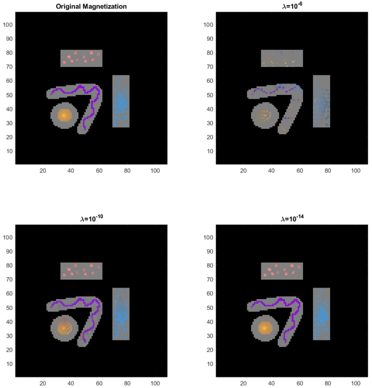


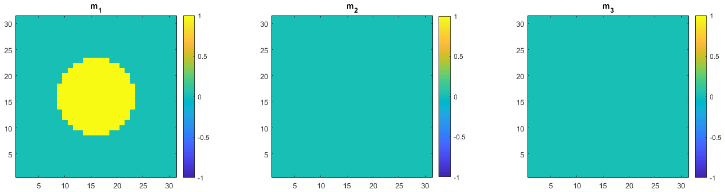
Illustration for Unidirectional recovery





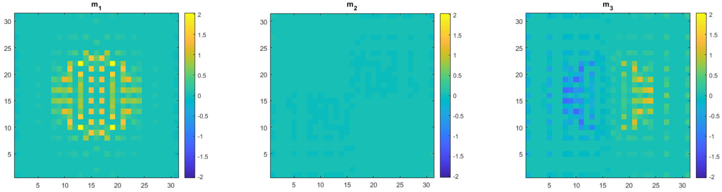
Unidirectional recovery in detail

Original



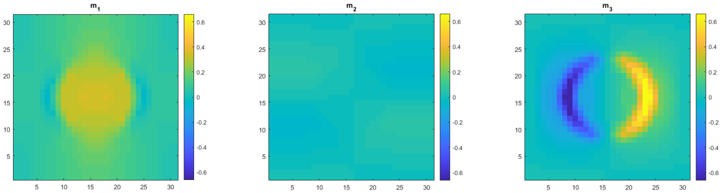
Unidirectional recovery in detail

l1 with 6e-4 relative residual



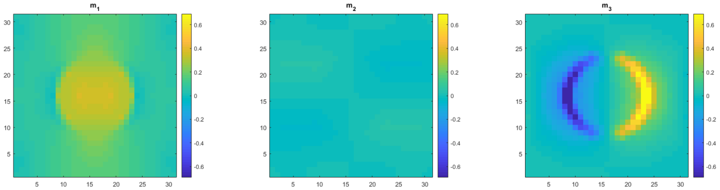
Unidirectional recovery in detail

TV with 4e-4 relative residual (separated norm)



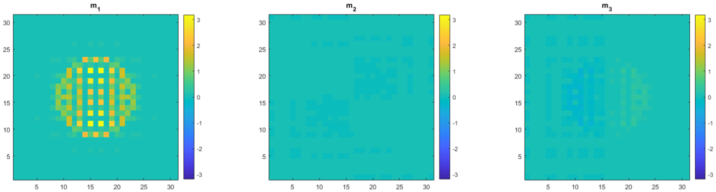
Unidirectional recovery in detail

TV with 4e-5 relative residual (l2 norm of components)



Unidirectional recovery in detail

Simulating full measurements, 5e-3 residual.



Thank you

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