FEM-BEM coupling for multi-domain acoustic scattering problems

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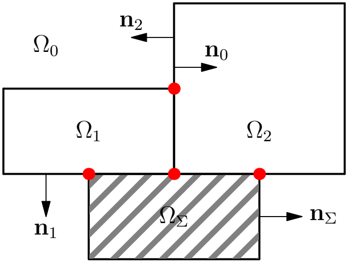
Multi-domain acoustic scattering

The transmission problem

Find $U \in \mathrm{H}^1_{\text{loc}}(\mathbb{R}^d)$ s.t.

$$\begin{cases} -\Delta U - \kappa_\Sigma^2(\mathbf{x})U = f \text{ in } \Omega_\Sigma \\ -\Delta U - \kappa_j^2 U = 0 \text{ in } \Omega_j,\ j = 0, \dots, n \\ U - U_{\text{inc}} \text{ outgoing in } \Omega_0 \\ \left.\begin{array}{l} \gamma_{\mathrm{D}}^j U - \gamma_{\mathrm{D}}^k U = 0 \\ \gamma_{\mathrm{N}}^j U + \gamma_{\mathrm{N}}^k U = 0 \end{array}\right. \text{ on } \partial\Omega_j \cap \partial\Omega_k,\ j \neq k \\ \left.\begin{array}{l} \gamma_{\mathrm{D}}^j U - \gamma_{\mathrm{D}}^\Sigma U = 0 \\ \gamma_{\mathrm{N}}^j U + \gamma_{\mathrm{N}}^\Sigma U = 0 \end{array}\right. \text{ on } \partial\Omega_j \cap \partial\Omega_\Sigma \end{cases}$$



Trace operators (by density)

$\gamma_{\mathrm{D}}^j \varphi := \varphi|_{\partial\Omega_j}$

$\gamma_{\mathrm{N}}^j \varphi := \mathbf{n}_j \cdot \nabla\varphi|_{\partial\Omega_j}$

⚠ Junction points ⚠

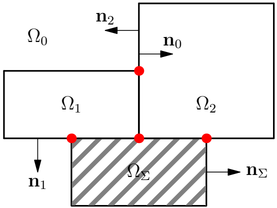
Goal: stable formulation with FEM in Ω_Σ + BEM in Ω_j, $j = 0, \dots, n$

FEM vs BEM: benefit from both!

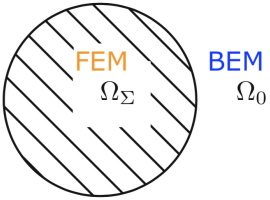
Finite Element Method (FEM): discretization of a partial differential equation

Boundary Element Method (BEM): discretization of a Boundary Integral Equation (BIE): reformulation on the boundary of the object

- surface mesh ⇒ reduction in the number of unknowns,
- no need of artificial boundary conditions for an unbounded domain,
- non local integral kernels ⇒ dense matrices,
- possible only if piecewise constant propagation medium.



FEM-BEM coupling strategies (2 domains)



- **Johnson-Nédélec coupling**: direct BIE in Ω_0
- **Costabel coupling**: direct BIE in Ω_0, symmetric!
- **Bielak-MacCamy coupling**: indirect BIE in Ω_0

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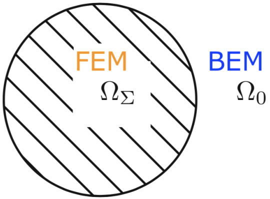

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Boundary integral formulations for multi-domain setting

- **Single-Trace Formulation** (STF) (also called Rumsey or PMCHWT): *only one* pair of traces at each point of each interface

  [Rumsey, 1954], [Poggio & Miller, 1973], [Chang & Harrington, 1977], [VonPetersdorff, 1989] ...

- **Boundary Element Tearing and Interconnecting method** (BETI): boundary integral equation version of FETI

  [Langer & Steinbach, 2003], [Langer & al, 2007], [Steinbach & Of, 2009] ...

- STF of **2nd kind**: multi-domain version of Müller formulation with non-singular integral kernels

  [Claeys, 2011], [Greengard & Lee, 2012], [Claeys, Hiptmair & Spindler, 2017]

- **Multi-Trace Formulation** (MTF) (global, quasi-local, local): pair of traces are *doubled* on each interface

  [Hiptmair & Jerez-Hanckes, 2012], [Claeys & Hiptmair, 2013], [Claeys, 2015] ...

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Multi-domain FEM-BEM with junction points

Multi-domain coupling with junction points

- mainly explored strategy is FETI-BETI
- here we choose to study STF (or MTF) for the BEM part: proper functional spaces even with junction points

Outline

Pair of traces and potentials

$$\mathbb{H}(\partial\Omega_j) = \mathrm{H}^{1/2}(\partial\Omega_j) \times \mathrm{H}^{-1/2}(\partial\Omega_j)$$

Interior and exterior trace operators:

$$\gamma^j V = \begin{pmatrix} \gamma_{\mathrm{D}}^j V \\ \gamma_{\mathrm{N}}^j V \end{pmatrix} \in \mathbb{H}(\partial\Omega_j) \qquad \gamma_c^j V = \begin{pmatrix} \gamma_{\mathrm{D},c}^j V \\ \gamma_{\mathrm{N},c}^j V \end{pmatrix} \in \mathbb{H}(\partial\Omega_j)$$

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$\mathcal{G}_\kappa(\mathsf{x})$: Green kernel (or fundamental solution) for the Helmholtz operator

$$-\Delta\mathcal{G}_\kappa(\mathsf{x}) - \kappa^2\mathcal{G}_\kappa(\mathsf{x}) = \delta_0(\mathsf{x}) \quad \text{in } \mathbb{R}^d$$

$$\text{for } \kappa \text{ constant} \qquad \mathcal{G}_\kappa(\mathsf{x}) = \frac{\exp(\imath\kappa|\mathsf{x}|)}{4\pi|\mathsf{x}|} \quad \text{for } d = 3$$

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Potential operator (DL + SL potentials): for $\mathfrak{v}_j = \binom{v}{q} \in \mathbb{H}(\partial\Omega_j)$

$$\mathrm{G}^j_\kappa(\mathfrak{v}_j)(\mathbf{x}) = \int_{\partial\Omega_j} v(\mathbf{y})\, \mathbf{n}_j(\mathbf{y}) \cdot (\nabla \mathcal{G}_\kappa)(\mathbf{x}-\mathbf{y})\, d\sigma(\mathbf{y}) + \int_{\partial\Omega_j} q(\mathbf{y})\, \mathcal{G}_\kappa(\mathbf{x}-\mathbf{y})\, d\sigma(\mathbf{y})$$

Representation formula and Calderón identities

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Representation formula

If $U \in \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j)$ with $-\Delta U - \kappa^2 U = 0$ in Ω_j (outgoing if $j = 0$) then

$$U(\mathbf{x})\, \mathbf{1}_{\Omega_j}(\mathbf{x}) = \mathsf{G}^j_\kappa(\gamma^j U)(\mathbf{x})$$

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Take interior traces of repr. formula

Calderón identities

$$\gamma^j \circ \mathrm{G}^j_\kappa(\gamma^j U) = \gamma^j U$$

characterizes traces of solutions to homogeneous Helmholtz eq!

$\gamma^j \circ \mathrm{G}^j_\kappa$: Calderón projector

Calderón identities and boundary integral operators

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Boundary integral operators:

$$\mathsf{A}^j_\kappa = \{\gamma^j\} \circ \mathsf{G}^j_\kappa = \frac{1}{2}(\gamma^j + \gamma^j_c) \circ \mathsf{G}^j_\kappa = \begin{pmatrix} \mathsf{K}_j & \mathsf{V}_j \\ \mathsf{W}_j & \mathsf{K}'_j \end{pmatrix}$$

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By jump relations ...

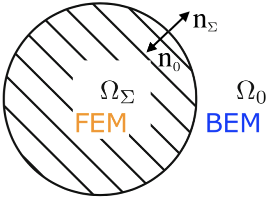
$$\gamma^j \circ \mathsf{G}^j_\kappa = \mathsf{A}^j_\kappa + \mathsf{Id}/2$$

so the characterization becomes

$$(\mathsf{A}^j_\kappa - \mathsf{Id}/2)(\gamma^j U) = 0$$

Classical Costabel coupling: 2 domains

$n = 0 \quad \mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_\Sigma \quad \Gamma = \partial\Omega_0 = \partial\Omega_\Sigma$



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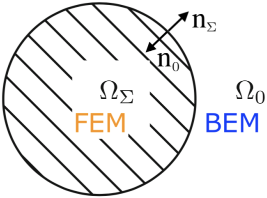

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In Ω_Σ: find $U \in \mathrm{H}^1(\Omega_\Sigma)$ s.t. $\quad \langle v, q\rangle_\Gamma = \int_\Gamma vq\, d\sigma$

$$\int_{\Omega_\Sigma} (\nabla U \cdot \nabla V - \kappa_\Sigma^2(\mathbf{x}) UV)\, d\mathbf{x} - \left\langle \gamma_{\mathrm{D}}^\Sigma V, \gamma_{\mathrm{N}}^\Sigma U \right\rangle_\Gamma = \int_{\Omega_\Sigma} fV\, d\mathbf{x} \quad \forall V \in \mathrm{H}^1(\Omega_\Sigma)$$

using $\gamma_{\mathrm{N}}^0 U = -\gamma_{\mathrm{N}}^\Sigma U$:

$$a_\Sigma(U, V) + \left\langle \gamma_{\mathrm{D}}^\Sigma V, \gamma_{\mathrm{N}}^0 U \right\rangle_\Gamma = F_\Sigma(V) \quad \forall V \in \mathrm{H}^1(\Omega_\Sigma)$$

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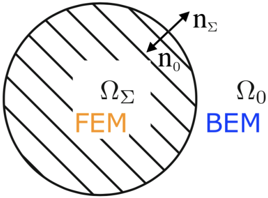

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Classical Costabel coupling: 2 domains

In Ω_0: *both* Calderón identities

$$\gamma^0(U - U_{\mathrm{inc}}) = \gamma^0 \mathsf{G}^0_{\kappa_0}(\gamma^0(U - U_{\mathrm{inc}}))$$

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using $\gamma^0_{\mathrm{D}} U = \gamma^\Sigma_{\mathrm{D}} U$, unknown $p := \gamma^0_{\mathrm{N}} U$.

Find $p \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$ s.t.

$$\left\langle \gamma^\Sigma_{\mathrm{D}} U, q \right\rangle_\Gamma - \left\langle \gamma^0_{\mathrm{D}} \mathsf{G}^0_{\kappa_0} \begin{pmatrix} \gamma^\Sigma_{\mathrm{D}} U \\ p \end{pmatrix}, q \right\rangle_\Gamma = \left\langle \gamma^0_{\mathrm{D}} U_{\mathsf{inc}}, q \right\rangle_\Gamma \quad \forall\, q \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$$

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Classical Costabel coupling: 2 domains

After subtracting and simplifying...
defining the skew-symmetric duality pairing for $\binom{u}{p}, \binom{v}{q} \in \mathbb{H}(\Gamma)$

$$\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}\right]_\Gamma := \langle u, q\rangle_\Gamma - \langle v, p\rangle_\Gamma$$

Costabel coupling

find $U \in \mathrm{H}^1(\Omega_\Sigma)$, $p \in \mathrm{H}^{-1/2}(\Gamma)$ such that

$$a_\Sigma(U, V) + \left[\mathsf{A}^0_{\kappa_0}\begin{pmatrix} \gamma_{\mathrm{D}}^{\Sigma} U \\ p \end{pmatrix}, \begin{pmatrix} -\gamma_{\mathrm{D}}^{\Sigma} V \\ q \end{pmatrix}\right]_\Gamma - \frac{1}{2}\left[\begin{pmatrix} \gamma_{\mathrm{D}}^{\Sigma} U \\ p \end{pmatrix}, \begin{pmatrix} \gamma_{\mathrm{D}}^{\Sigma} V \\ q \end{pmatrix}\right]_\Gamma$$
$$= F_\Sigma(V) - \left[\gamma^0 U_{\mathrm{inc}}, \begin{pmatrix} -\gamma_{\mathrm{D}}^{\Sigma} V \\ q \end{pmatrix}\right]_\Gamma \quad \forall\, V \in \mathrm{H}^1(\Omega_\Sigma),\ q \in \mathrm{H}^{-1/2}(\Gamma)$$

2×2 operator $\mathsf{A}^0_{\kappa_0}$: explicit classical boundary integral operators

Classical Costabel coupling: 2 domains

Generalized Gårding inequality

$\mathsf{a_C}$: the bilinear form of Costabel coupling.
There exist a compact bilinear form k, $\beta > 0$ s.t.
$\mathrm{Re}\left\{\mathsf{a_C}\big((V,q),(\overline{V},\overline{q})\big) + \mathsf{k}\big((V,q),(\overline{V},\overline{q})\big)\right\} \geq \beta(\|V\|^2_{\mathrm{H}^1(\Omega_\Sigma)} + \|q\|^2_{\mathrm{H}^{-1/2}(\Gamma)})$
for all $V \in \mathrm{H}^1(\Omega_\Sigma)$, $q \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$.

$\Rightarrow$ bijective if and only if injective!

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for all $V \in \mathrm{H}^1(\Omega_\Sigma)$, $q \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$.

$\Rightarrow$ bijective if and only if injective!

Spurious resonances

Let $U \in \mathrm{H}^1(\Omega_\Sigma)$, $p \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$ solve Costabel formulation with $f = 0$, $U_{\mathrm{inc}} = 0$.
Then

- $U = 0$
- $p = 0$ IFF κ_0^2 is not an interior Dirichlet eigenvalue of $-\Delta$ on Ω_Σ, i.e.

$$\nexists\, W \in \mathrm{H}^1(\Delta, \Omega_\Sigma)\backslash\{0\} \text{ such that } \begin{aligned} -\Delta W &= \kappa_0^2 W && \text{in } \Omega_\Sigma \\ W &= 0 && \text{on } \partial\Omega_\Sigma. \end{aligned}$$

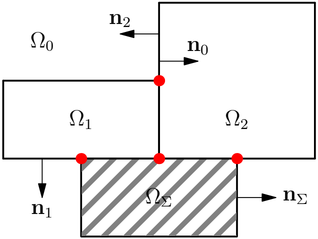
See well-posed CFIE formulation in [Hiptmair & Meury, 2006]

Outline

Trace spaces for multi-domain scattering

$\mathbb{R}^d = \bigcup_{j=0}^{n} \overline{\Omega}_j \cup \overline{\Omega}_\Sigma$

$\Gamma = \bigcup_{j=0}^{n} \partial\Omega_j$ skeleton $\Sigma = \partial\Omega_\Sigma$



$\mathbb{H}(\partial\Omega_j) = \mathrm{H}^{+1/2}(\partial\Omega_j) \times \mathrm{H}^{-1/2}(\partial\Omega_j)$

for $\mathfrak{u}_j = (u, p)$, $\mathfrak{v}_j = (v, q) \in \mathbb{H}(\partial\Omega_j)$

$[\mathfrak{u}_j, \mathfrak{v}_j]_j = \langle u, q \rangle_j - \langle v, p \rangle_j$

Trace spaces for multi-domain scattering

$\mathbb{R}^d = \bigcup_{j=0}^n \overline{\Omega}_j \cup \overline{\Omega}_\Sigma$

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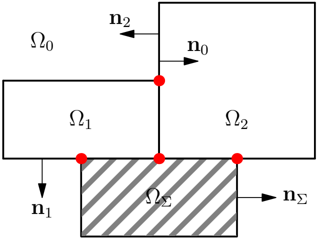

Multi-trace space: $\mathbb{H}(\Gamma) = \mathbb{H}(\partial\Omega_0) \times \cdots \times \mathbb{H}(\partial\Omega_n)$

Duality pairing:

for $\mathfrak{u} = (\mathfrak{u}_0, \ldots, \mathfrak{u}_n), \mathfrak{v} = (\mathfrak{v}_0, \ldots, \mathfrak{v}_n) \in \mathbb{H}(\Gamma)$ $[\mathfrak{u}, \mathfrak{v}] = \sum_{j=0}^n [\mathfrak{u}_j, \mathfrak{v}_j]_j$

Trace spaces for multi-domain scattering

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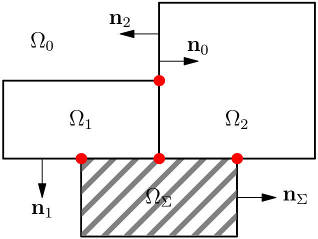

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Single-trace space $\mathbb{X}(\Gamma) \subset \mathbb{H}(\Gamma)$:

$\mathbb{X}(\Gamma)$ = tuples of traces that match the transmission conditions

Trace spaces for multi-domain scattering

$\mathbb{R}^d = \bigcup_{j=0}^n \overline{\Omega}_j \cup \overline{\Omega}_\Sigma$
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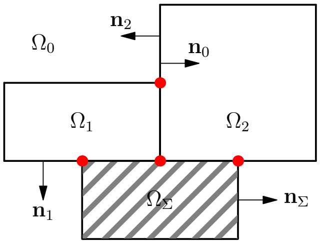

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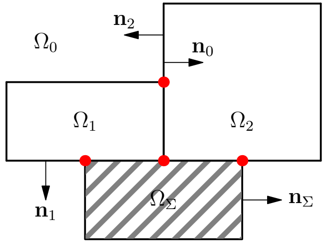
Single-trace space $\mathbb{X}(\Gamma) \subset \mathbb{H}(\Gamma)$:
$\mathbb{X}(\Gamma)$ = tuples of traces that match the transmission conditions
$\mathbb{X}(\Gamma) = \{ \mathfrak{u} = (u_j, p_j)_{j=0}^n \mid (u_j)_{j=0}^n \in \mathbb{X}^{+1/2}(\Gamma),\ (p_j)_{j=0}^n \in \mathbb{X}^{-1/2}(\Gamma) \}$
$\mathbb{X}^{+1/2}(\Gamma) = \{ (V|_{\partial\Omega_j})_{j=0}^n \mid V \in \mathrm{H}^1(\mathbb{R}^d) \}$
$\mathbb{X}^{-1/2}(\Gamma) = \{ (\mathbf{n}_j \cdot \mathbf{q}|_{\partial\Omega_j})_{j=0}^n \mid \mathbf{q} \in \mathrm{H}^1(\mathrm{div}, \mathbb{R}^d) \}$

Trace spaces for multi-domain scattering



Every $\mathbf{x} \in \Sigma = \partial\Omega_\Sigma$ also belongs to some $\partial\Omega_j$, $j = 0, \ldots, n$

Proposition [Claeys & Hiptmair, 2015]

A tuple in $\mathbb{X}(\Gamma)$ induces unique traces in $\mathbb{H}(\Sigma)$.
The resulting operator $\mathsf{T}\colon \mathbb{X}(\Gamma) \to \mathbb{H}(\Sigma)$, $\mathfrak{u} \mapsto (\mathsf{T}_{\mathrm{D}}(\mathfrak{u}), \mathsf{T}_{\mathrm{N}}(\mathfrak{u}))$
is continuous and surjective.

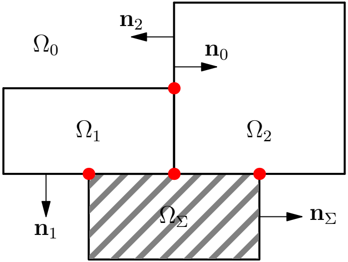
For all $\mathfrak{u}$, $\mathfrak{v} \in \mathbb{X}(\Gamma)$ we have $\quad [\mathfrak{u}, \mathfrak{v}] = -[\mathsf{T}(\mathfrak{u}), \mathsf{T}(\mathfrak{v})]_\Sigma$

[Claeys & Hiptmair, *Integral Equations for Acoustic Scattering by Partially Impenetrable Composite Objects*, 2015]

Single-Trace FEM-BEM formulation

Goal: stable formulation with FEM in Ω_Σ + BEM in Ω_j, $j = 0, \ldots, n$ for the transmission problem

$$\begin{cases} -\Delta U - \kappa_\Sigma^2(\mathsf{x})U = f \text{ in } \Omega_\Sigma \\ -\Delta U - \kappa_j^2 U = 0 \text{ in } \Omega_j,\ j = 0, \ldots, n \\ U - U_{\text{inc}} \text{ outgoing in } \Omega_0 \\ \left.\begin{array}{l} \gamma_D^j U - \gamma_D^k U = 0 \\ \gamma_N^j U + \gamma_N^k U = 0 \end{array}\right\} \text{ on } \partial\Omega_j \cap \partial\Omega_k,\ j \neq k \\ \left.\begin{array}{l} \gamma_D^j U - \gamma_D^\Sigma U = 0 \\ \gamma_N^j U + \gamma_N^\Sigma U = 0 \end{array}\right\} \text{ on } \partial\Omega_j \cap \Sigma \end{cases}$$



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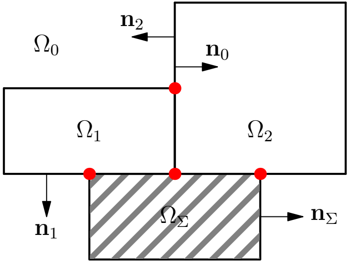
$$



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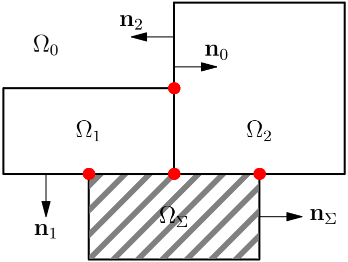

Single-Trace FEM-BEM formulation

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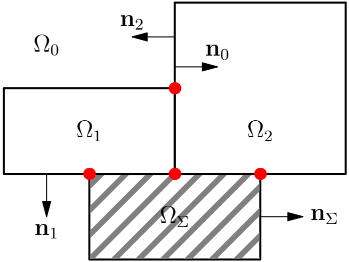

Single-Trace FEM-BEM formulation

First reformulate the piece-wise homogenous part by BIEs

$$\begin{cases} \left.\begin{array}{l} \gamma_{\mathrm{D}}^{j} U - \gamma_{\mathrm{D}}^{k} U = 0 \\ \gamma_{\mathrm{N}}^{j} U + \gamma_{\mathrm{N}}^{k} U = 0 \end{array}\right. \text{ on } \partial\Omega_j \cap \partial\Omega_k, j \neq k \\ \gamma_{\mathrm{D}}^{j} U - \gamma_{\mathrm{D}}^{\Sigma} U = 0 \ \text{ on } \partial\Omega_j \cap \Sigma \end{cases}$$

Set $\mathfrak{u} := (\gamma^0 U, \ldots, \gamma^n U)$

Seek
$\mathfrak{u} \in \mathbb{X}(\Gamma)$ with $\gamma_{\mathrm{D}}^{\Sigma} U = \mathsf{T}_{\mathrm{D}}(\mathfrak{u})$

Single-Trace FEM-BEM formulation

First reformulate the piece-wise homogenous part by BIEs

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Seek
$\mathfrak{u} \in \mathbb{X}(\Gamma)$ with $\gamma_{\mathrm{D}}^{\Sigma} U = \mathsf{T}_{\mathrm{D}}(\mathfrak{u})$

$$\begin{cases} -\Delta U - \kappa_j^2 U = 0 \text{ in } \Omega_j,\ j = 0, \ldots, n \\ U - U_{\mathrm{inc}} \text{ outgoing in } \Omega_0 \end{cases}$$

$$(\mathsf{A} - \mathsf{Id}/2)(\mathfrak{u} - \mathfrak{u}^{\mathrm{inc}}) = 0$$

where $\mathfrak{u}^{\mathrm{inc}} := (\gamma^0 U_{\mathrm{inc}}, 0, \ldots, 0)$ and $\mathsf{A} \colon \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$

$$\mathsf{A}(\mathfrak{u}) := \left(\mathsf{A}^j_{\kappa_j}(\mathfrak{u}_j)\right)_{j=0}^{n} = \begin{bmatrix} \mathsf{A}^0_{\kappa_0} & 0 & \ldots & 0 \\ 0 & \mathsf{A}^1_{\kappa_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \mathsf{A}^n_{\kappa_n} \end{bmatrix} \begin{bmatrix} \mathfrak{u}_0 \\ \vdots \\ \vdots \\ \mathfrak{u}_n \end{bmatrix}$$

Single-Trace FEM-BEM formulation

$\mathfrak{u} \in \mathbb{X}(\Gamma)$ with $\gamma_{\mathrm{D}}^{\Sigma} U = \mathsf{T}_{\mathrm{D}}(\mathfrak{u})$ such that

$$(\mathsf{A} - \mathsf{Id}/2)\mathfrak{u} = -\mathfrak{u}^{\mathsf{inc}}$$

variational form: $\forall\, \mathfrak{v} \in \mathbb{X}(\Gamma)$ with $\mathsf{T}_{\mathrm{D}}(\mathfrak{v}) = \gamma_{\mathrm{D}}^{\Sigma} V$

$$[\mathsf{A}(\mathfrak{u}), \mathfrak{v}] \quad \underbrace{-\frac{1}{2}[\mathfrak{u}, \mathfrak{v}]}_{\substack{\| \\ +\frac{1}{2}[\mathsf{T}(\mathfrak{u}),\mathsf{T}(\mathfrak{v})]_{\Sigma}}} \quad = -[\mathfrak{u}^{\mathsf{inc}}, \mathfrak{v}]$$

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variational form: $\quad \forall\, \mathfrak{v} \in \mathbb{X}(\Gamma)$ with $\mathsf{T}_{\mathrm{D}}(\mathfrak{v}) = \gamma_{\mathrm{D}}^{\Sigma} V$

$$[\mathsf{A}(\mathfrak{u}), \mathfrak{v}] + \frac{1}{2}[\mathsf{T}(\mathfrak{u}), \mathsf{T}(\mathfrak{v})]_{\Sigma} = -[\mathfrak{u}^{\mathsf{inc}}, \mathfrak{v}]$$

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$$[\mathsf{A}(\mathfrak{u}), \Theta(\mathfrak{v})] + \frac{1}{2}[\mathsf{T}(\mathfrak{u}), \mathsf{T}(\Theta(\mathfrak{v}))]_{\Sigma} = -[\mathfrak{u}^{\mathsf{inc}}, \Theta(\mathfrak{v})]$$

where

$$\Theta_j \begin{pmatrix} v_j \\ q_j \end{pmatrix} := \begin{pmatrix} -v_j \\ q_j \end{pmatrix}, \quad \Theta(\mathfrak{v}) := \left(\Theta_j(\mathfrak{v}_j)\right)_{j=0}^{n} \quad \text{for } \mathfrak{v} = \begin{pmatrix} v_j \\ q_j \end{pmatrix}_{j=0}^{n} \in \mathbb{H}(\Gamma)$$

Single-Trace FEM-BEM formulation

$\mathfrak{u} \in \mathbb{X}(\Gamma)$ with $\gamma_{\mathrm{D}}^{\Sigma} U = \mathsf{T}_{\mathrm{D}}(\mathfrak{u})$ such that

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Single-Trace FEM-BEM formulation

Now reformulate the heterogenous part by usual domain var. form.

$$\begin{cases} -\Delta U - \kappa_\Sigma^2(\mathsf{x})U = f \text{ in } \Omega_\Sigma \\ \gamma_{\mathrm{N}}^j U + \gamma_{\mathrm{N}}^\Sigma U = 0 \ \text{ on } \partial\Omega_j \cap \Sigma,\ j = 0, \dots, n \end{cases}$$

In Ω_Σ: find $U \in \mathrm{H}^1(\Omega_\Sigma)$ s.t.

$$\int_{\Omega_\Sigma} (\nabla U \cdot \nabla V - \kappa_\Sigma^2(\mathsf{x}) UV)\, d\mathsf{x} - \left\langle \gamma_{\mathrm{D}}^\Sigma V, \gamma_{\mathrm{N}}^\Sigma U \right\rangle_\Sigma = \int_{\Omega_\Sigma} fV\, d\mathsf{x} \quad \forall\, V \in \mathrm{H}^1(\Omega_\Sigma)$$

using $\gamma_{\mathrm{N}}^\Sigma U = +\mathsf{T}_{\mathrm{N}}(\mathfrak{u})$:

$$a_\Sigma(U, V) - \left\langle \gamma_{\mathrm{D}}^\Sigma V, \mathsf{T}_{\mathrm{N}}(\mathfrak{u}) \right\rangle_\Sigma = F_\Sigma(V) \quad \forall\, V \in \mathrm{H}^1(\Omega_\Sigma)$$

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$$\int_{\Omega_\Sigma} (\nabla U \cdot \nabla V - \kappa_\Sigma^2(\mathsf{x}) UV)\, d\mathsf{x} - \left\langle \gamma_{\mathrm{D}}^\Sigma V, \gamma_{\mathrm{N}}^\Sigma U \right\rangle_\Sigma = \int_{\Omega_\Sigma} fV\, d\mathsf{x} \quad \forall\, V \in \mathrm{H}^1(\Omega_\Sigma)$$

using $\gamma_{\mathrm{N}}^\Sigma U = +\mathsf{T}_{\mathrm{N}}(\mathfrak{u})$:

$$a_\Sigma(U, V) - \left\langle \gamma_{\mathrm{D}}^\Sigma V, \mathsf{T}_{\mathrm{N}}(\mathfrak{u}) \right\rangle_\Sigma = F_\Sigma(V) \quad \forall\, V \in \mathrm{H}^1(\Omega_\Sigma)$$

Single-Trace FEM-BEM formulation

Summing and simplifying

Single-Trace FEM-BEM formulation

find $U \in \mathrm{H}^1(\Omega_\Sigma)$, $\mathfrak{u} \in \mathbb{X}(\Gamma)$, with $\gamma_{\mathrm{D}}^{\Sigma} U = \mathsf{T}_{\mathrm{D}}(\mathfrak{u})$ such that

$$a_\Sigma(U, V) + [\mathsf{A}(\mathfrak{u}), \Theta(\mathfrak{v})] + \frac{1}{2}\left[\begin{pmatrix}\gamma_{\mathrm{D}}^{\Sigma} U \\ \mathsf{T}_{\mathrm{N}}(\mathfrak{u})\end{pmatrix}, \begin{pmatrix}\gamma_{\mathrm{D}}^{\Sigma} V \\ \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\end{pmatrix}\right]_\Sigma$$

$$= F_\Sigma(V) - [\mathfrak{u}^{\mathrm{inc}}, \Theta(\mathfrak{v})] \quad \forall\, V \in \mathrm{H}^1(\Omega_\Sigma),\ \mathfrak{v} \in \mathbb{X}(\Gamma) \text{ with } \gamma_{\mathrm{D}}^{\Sigma} V = \mathsf{T}_{\mathrm{D}}(\mathfrak{v})$$

Summing and simplifying

Single-Trace FEM-BEM formulation

find $U \in \mathrm{H}^1(\Omega_\Sigma)$, $\mathfrak{u} \in \mathbb{X}(\Gamma)$, with $\gamma_{\mathrm{D}}^{\Sigma} U = \mathsf{T}_{\mathrm{D}}(\mathfrak{u})$ such that

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$$= F_\Sigma(V) - [\mathfrak{u}^{\mathrm{inc}}, \Theta(\mathfrak{v})] \quad \forall\, V \in \mathrm{H}^1(\Omega_\Sigma),\ \mathfrak{v} \in \mathbb{X}(\Gamma) \text{ with } \gamma_{\mathrm{D}}^{\Sigma} V = \mathsf{T}_{\mathrm{D}}(\mathfrak{v})$$

Proposition (Representation formula)

The solution to the transmission problem is given by

$$\widetilde{U}(\mathbf{x}) := U(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_\Sigma,$$

$$\widetilde{U}(\mathbf{x}) := \left(\mathsf{G}^0_{\kappa_0}(\mathfrak{u}_0) + U_{\mathrm{inc}}\right)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_0,$$

$$\widetilde{U}(\mathbf{x}) := \mathsf{G}^j_{\kappa_j}(\mathfrak{u}_j)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_j, j = 1, \ldots, n.$$

Single-Trace FEM-BEM formulation

A generalized Gårding inequality is satisfied

Consequences: *in the case of injectivity*

- well-posedness
- stability (inf-sup condition)
- for Galerkin, discrete inf-sup condition and quasi-optimal convergence

Single-Trace FEM-BEM formulation

A generalized Gårding inequality is satisfied

Consequences: *in the case of injectivity*

- well-posedness
- stability (inf-sup condition)
- for Galerkin, discrete inf-sup condition and quasi-optimal convergence

Proposition (Injectivity condition)

Let $U \in \mathrm{H}^1(\Omega_\Sigma)$, $\mathfrak{u} \in \mathbb{X}(\Gamma)$ with $\gamma_{\mathrm{D}}^{\Sigma} U = \mathsf{T}_{\mathrm{D}}(\mathfrak{u})$ solve the FEM-BEM STF with $f = 0$, $U_{\mathsf{inc}} = 0$. Then

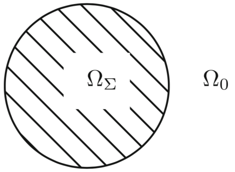
- $U = 0$
- $\mathfrak{u} = 0$ is the unique solution IFF

$$\text{for all } j = 0, \dots, n, \quad \Sigma \not\subset \partial\Omega_j \ \text{ or } \ \kappa_j \notin \mathfrak{S}(\Delta, \Omega_\Sigma)$$

(i.e. κ_j^2 not an interior Dirichlet eigenvalue of $-\Delta$ on Ω_Σ)

Costabel coupling: $\Sigma \subset \partial\Omega_0$

$\Rightarrow$ spurious resonances if $\kappa_0 \in \mathfrak{S}(\Delta, \Omega_\Sigma)$



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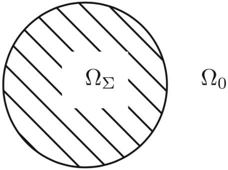
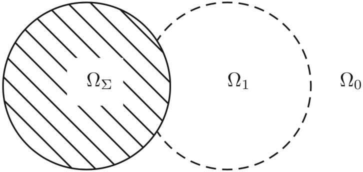


$\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \overline{\Omega}_\Sigma$, but $\kappa_0 = \kappa_1$

$\Sigma \not\subset \partial\Omega_1$ and $\Sigma \not\subset \partial\Omega_0$

$\Rightarrow$ no spurious resonances no matter κ_0!



Results analogue to

[Claeys & Hiptmair, *Integral Equations for Acoustic Scattering by Partially Impenetrable Composite Objects*, 2015]

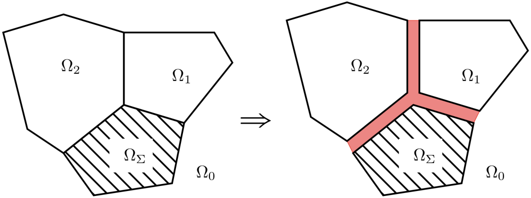
Outline

Gap setting and new trace spaces

Problem with STF: $\mathbb{X}(\Gamma)$ contains transmission conditions in strong form

- not flexible
- obstacle to operator preconditioning [Claeys, Hiptmair & Jerez-Hanckes, 2013]

$\Rightarrow$ Multi-Trace Formulations:



idea: apply STFs to gap configurations with vanishing gap

Gap setting and new trace spaces

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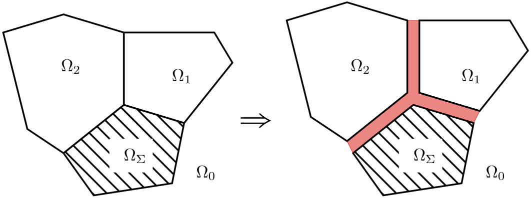


idea: apply STFs to gap configurations with vanishing gap

$\{ (U, \mathfrak{u}) \in \mathrm{H}^1(\Omega_\Sigma) \times \mathbb{X}(\Gamma) \mid \gamma_{\mathrm{D}}^{\Sigma} U = \mathsf{T}_{\mathrm{D}}(\mathfrak{u}) \}$ isomorphic to $\mathrm{H}^1(\Omega_\Sigma) \times \widehat{\mathbb{H}}(\Gamma)$

multi-trace space:

$$\widehat{\mathbb{H}}(\Gamma) = \mathbb{H}(\partial\Omega_1) \times \cdots \times \mathbb{H}(\partial\Omega_n) \times \mathrm{H}^{-\frac{1}{2}}(\Sigma)$$

Multi-Trace FEM-BEM formulation

Elaborate the FEM-BEM STF by eliminating all the contributions on $\partial\Omega_0$...

Global Multi-Trace FEM-BEM formulation

find $U \in \mathrm{H}^1(\Omega_\Sigma)$, $\hat{\mathfrak{u}} \in \widehat{\mathbb{H}}(\Gamma), \hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_1, \ldots, \hat{\mathfrak{u}}_n, p_\Sigma)$, such that

$$a_\Sigma(U, V) + \{\!\![\widehat{\widehat{\mathsf{A}}}(\hat{\hat{\mathfrak{u}}}), \Theta(\hat{\hat{\mathfrak{v}}})]\!\!\} + \frac{1}{2}\left[\begin{pmatrix}\gamma_{\mathrm{D}}^{\Sigma} U \\ p_\Sigma\end{pmatrix}, \begin{pmatrix}\gamma_{\mathrm{D}}^{\Sigma} V \\ q_\Sigma\end{pmatrix}\right]_\Sigma$$

$$= F_\Sigma(V) + \{\!\![\hat{\hat{\mathfrak{f}}}, \Theta(\hat{\hat{\mathfrak{v}}})]\!\!\} \quad \forall\, V \in \mathrm{H}^1(\Omega_\Sigma),\ \hat{\mathfrak{v}} \in \widehat{\mathbb{H}}(\Gamma), \hat{\mathfrak{v}} = (\hat{\mathfrak{v}}_1, \ldots, \hat{\mathfrak{v}}_n, q_\Sigma)$$

where $\hat{\hat{\mathfrak{u}}} := (\hat{\mathfrak{u}}_1, \ldots, \hat{\mathfrak{u}}_n, (\gamma_{\mathrm{D}}^{\Sigma} U, p_\Sigma)) \quad \hat{\hat{\mathfrak{v}}} := (\hat{\mathfrak{v}}_1, \ldots, \hat{\mathfrak{v}}_n, (\gamma_{\mathrm{D}}^{\Sigma} V, q_\Sigma))$

$$\hat{\hat{\mathfrak{f}}} := (\gamma^1 U_{\mathrm{inc}}, \ldots, \gamma^n U_{\mathrm{inc}}, \gamma^\Sigma U_{\mathrm{inc}})$$

Skew-symmetric duality pairing: difference with $[\cdot,\cdot]$

$$\{\!\![\mathfrak{u}, \mathfrak{v}]\!\!\} := \sum_{j=1}^{n} [\mathfrak{u}_j, \mathfrak{v}_j]_j + [\mathfrak{u}_\Sigma, \mathfrak{v}_\Sigma]_\Sigma$$

Multi-Trace FEM-BEM formulation

$$\widehat{\widehat{\mathsf{A}}} = \begin{bmatrix} \mathsf{A}^1_{\kappa_1} + \mathsf{A}^1_{\kappa_0} & \gamma^1 \mathsf{G}^2_{\kappa_0} & \dots & \gamma^1 \mathsf{G}^n_{\kappa_0} & \gamma^1 \mathsf{G}^\Sigma_{\kappa_0} \\ \gamma^2 \mathsf{G}^1_{\kappa_0} & \mathsf{A}^2_{\kappa_2} + \mathsf{A}^2_{\kappa_0} & & \gamma^2 \mathsf{G}^n_{\kappa_0} & \gamma^2 \mathsf{G}^\Sigma_{\kappa_0} \\ \vdots & & \ddots & & \vdots \\ \gamma^n \mathsf{G}^1_{\kappa_0} & \gamma^n \mathsf{G}^2_{\kappa_0} & & \mathsf{A}^n_{\kappa_n} + \mathsf{A}^n_{\kappa_0} & \gamma^n \mathsf{G}^\Sigma_{\kappa_0} \\ \gamma^\Sigma \mathsf{G}^1_{\kappa_0} & \gamma^\Sigma \mathsf{G}^2_{\kappa_0} & \dots & \gamma^\Sigma \mathsf{G}^n_{\kappa_0} & \mathsf{A}^\Sigma_{\kappa_0} \end{bmatrix}$$

global MTF: all subdomains coupled with all other subdomains

Multi-Trace FEM-BEM formulation

Proposition (Representation formula)

The solution to the transmission problem is given by

$$\widetilde{U}(\mathsf{x}) := U(\mathsf{x}) \quad \text{for } \mathsf{x} \in \Omega_\Sigma$$

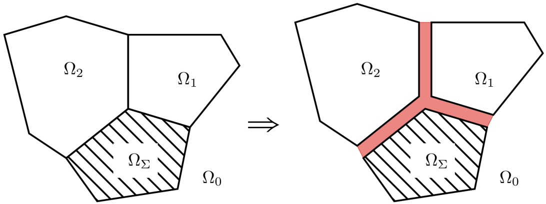
$$\widetilde{U}(\mathsf{x}) := \left(U_{\text{inc}} - \mathsf{G}^{\Sigma}_{\kappa_0} \begin{pmatrix} \gamma^{\Sigma}_{\mathrm{D}} U \\ p_\Sigma \end{pmatrix} - \sum_{j=1}^{n} \mathsf{G}^{j}_{\kappa_0}(\hat{\mathrm{u}}_j) \right)(\mathsf{x}) \quad \text{for } \mathsf{x} \in \Omega_0$$

$$\widetilde{U}(\mathsf{x}) := \mathsf{G}^{j}_{\kappa_j}(\hat{\mathrm{u}}_j)(\mathsf{x}) \quad \text{for } \mathsf{x} \in \Omega_j, j = 1, \ldots, n$$

Proposition

The formulation satisfies a generalized Gårding inequality

Multi-Trace FEM-BEM formulation



Gap configuration: $\Sigma \subset \partial\Omega_0 \Rightarrow$ spurious resonances expected!

Proposition (Injectivity condition)

Let $U \in \mathrm{H}^1(\Omega_\Sigma)$, $\hat{\mathfrak{u}} \in \widehat{\mathbb{H}}(\Gamma)$ solve the FEM-BEM MTF with $f = 0$, $U_{\text{inc}} = 0$. Then

- $U = 0$
- $\hat{\mathfrak{u}} = 0$ is the unique solution IFF $\kappa_0 \notin \mathfrak{S}(\Delta, \Omega_\Sigma)$

Conclusion and outlook

- Stable FEM-BEM formulations for multi-domain acoustic scattering, with junction points
- Generalized Gårding inequalities, injectivity conditions

Combined field versions immune to spurious resonances! ✓

Future work:

- implementation and numerical tests
- piece-wise constant coefficient in Neumann transmission conditions
- local FEM-BEM MTFs $\rightarrow$ optimized Schwarz methods
- quasi-local FEM-BEM MTFs
- other FEM-BEM coupling strategies
- FEM-BEM preconditioners