A combined inversion method for a geometrical inverse problem

Slim Chaabane[1], **Houssem Haddar**[2], **Rahma Jerbi**[1,*]

[1] University of Sfax.

[2] INRIA & Institut Polytechnique de Paris.

*Email: `rahma.jerbi.etud@fss.usf.tn`

Abstract

We consider the inverse geometrical problem of identifying the discontinuity curve of an electrical conductivity from boundary measurements. This standard inverse problem is used as a model to introduce and study a combined inversion algorithm coupling a gradient descent on the Kohn-Vogelius cost functional with a domain decomposition method that includes the unknown curve in the domain partitioning. We prove the local convergence of the method in a simplified case and numerically show its efficiency for some two dimensional experiments.

1. Motivations-Physical problem

- Let Ω be a simply connected bounded domain of $\mathbb{R}^2$ with $\mathcal{C}^{1,\beta}$ boundary $\Gamma := \partial\Omega$, $\beta \in]0,1[$.
- $\mathcal{S}_{ad} := \left\{ \sigma = \sigma_1 \chi_{\Omega_1} + \sigma_2 \chi_{\Omega_2}, \sigma_i > 0, \Sigma = \overline{\Omega_1} \cap \overline{\Omega_2} \text{ a } \mathcal{C}^{1,\beta} \right\}$

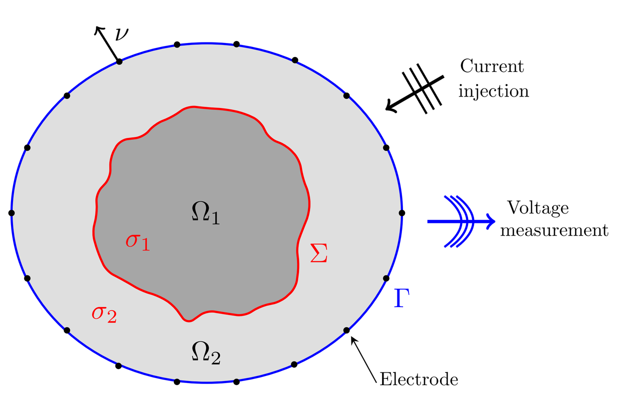


Figure 1: The domain $\Omega := \Omega_1 \cup \Omega_2 \cup \Sigma$.

Let $\phi \in \mathrm{L}^2(\Gamma)$ be the current injection through Γ and $\sigma \in \mathcal{S}_{ad}$, the electric conductivity, then the resulting electric potential u satisfies the following Neumann boundary value problem

$$(\mathcal{P}) \begin{cases} -\mathrm{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ \sigma \dfrac{\partial u}{\partial \nu} = \phi & \text{on } \Gamma. \end{cases}$$

▶ ϕ satisfies the following compatibility condition

$$\int_\Gamma \phi \, ds = 0.$$

▶ **Direct problem**: Known σ and ϕ, determine u.

▶ **Inverse problem**: Known ϕ and $f = u_{|\Gamma}$, determine σ.

▶ $(\mathcal{P})$ is almost always used for the modelling of the forward map in Electrical Impedance Tomography (EIT)

▶ **Some applications**: in medical imaging, geophysical prospection...

Electrical coring

Our idea

A combined inversion method for identifying the discontinuity curve Σ of σ from boundary measurements by combining a gradient descent method with a non-overlapping DDM.

2. The Kohn-Vogelius cost functional

$$J_{KV} : \mathcal{S}_{ad} \longrightarrow \mathbb{R}_+ \qquad \sigma \longmapsto \int_\Omega \sigma |\nabla w_\sigma|^2 \qquad w_\sigma = u_\sigma - v_\sigma,$$

u_σ and v_σ are respectively the solutions of (N_σ) and (D_σ)

$$(N_\sigma) \begin{cases} -\mathrm{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ \sigma \dfrac{\partial u}{\partial \nu} = \phi & \text{on } \Gamma. \\ \displaystyle\int_\Sigma u = 0. \end{cases} \qquad (D_\sigma) \begin{cases} -\mathrm{div}(\sigma \nabla v) = 0 & \text{in } \Omega, \\ v = f & \text{on } \Gamma. \end{cases}$$

▶ Differentiability of J_{KV} with respect to the discontinuity surface Σ of σ is provided by L. Afraites, M. Dambrine and D. Kateb in [1]

The gradient descent algorithm in the case of star-like domains [2]:

- $n \in \mathbb{N}^*$ and $R = (R_0, \dots, R_{n-1}) \in (\mathbb{R}_+^*)^n$ with $R_n = R_0$.
- Let $\Sigma := \Sigma_R$, we define the interface operator $\mathcal{T}_n$ by:

$$\mathcal{T}_n : (\mathbb{R}_+^*)^n \longrightarrow \mathcal{C}^{1,\beta} \qquad R \longmapsto \Sigma_R = \mathcal{T}_n(R) := \bigcup_{i=0}^{n-1} S_i, \tag{1}$$

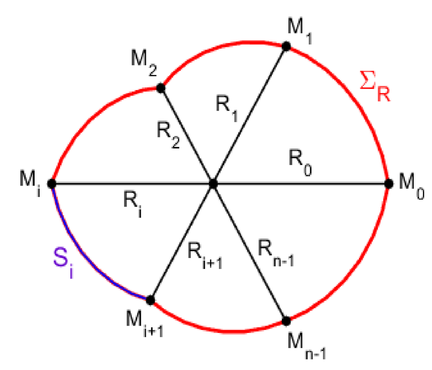


Figure 2: Description of the starlike geometry of $\Omega_1 := \Omega_{1,R}$.

▶ Let us set $\sigma(R) := \sigma_1 \chi_{\Omega_{1,R}} + \sigma_2 \chi_{\Omega_{2,R}}$ and define the function $\mathcal{J}$

$$\mathcal{J} : (\mathbb{R}_+^*)^n \longrightarrow \mathbb{R} \qquad R \longmapsto \mathcal{J}(R) = J_{KV}(\sigma(R)).$$

Algorithm 1: Gradient descent algorithm with exact direct solver

- Fix the number of parameters $n \in \mathbb{N}^*$ that serve to define the interface.
- Consider an initial guess $R^0 \in (\mathbb{R}_+^*)^n$ with $\Sigma = \Sigma_{R^0} = \mathcal{T}_n(R^0)$.
- $k = 0$.

repeat *until $k \leq$ maximum number of iterations*

- Use a direct solver to calculate $u_{\sigma(R^k)}$ and $v_{\sigma(R^k)}$.
- Calculate $\dfrac{\partial \mathcal{J}}{\partial R_i}(R^k)$, for $i = 0, \dots, n-1$.
- Update $\Sigma = \mathcal{T}_n(R^{k+1})$ with $R_i^{k+1} := R_i^k - \tau \dfrac{\partial \mathcal{J}}{\partial R_i}(R^k)$, where $\tau > 0$ is chosen sufficiently small.
- $R^k = R^{k+1}$.
- $k = k + 1$.

end repeat

3. Non-overlapping DDM

We present a nonoverlapping DDM (optimized Schwartz algorithm) for solving the direct problems (N_σ) and (D_σ) in the case where $\sigma \in \mathcal{S}_{ad}$.

Optimized Schwarz algorithm (see [4,3])

- Given an initial guess $\lambda_{i,N}^0 \in L^2(\Sigma)$ such that $\displaystyle\int_\Sigma \lambda_{i,N}^0 \, ds = 0, \; i = 1, 2$.
- For $\ell \geq 0$, **solve in parallel the Robin subdomain problems**:

$$\begin{cases} -\sigma_i \Delta u_i^{\ell+1} = 0 & \text{in } \Omega_i, i = 1, 2, \\ \sigma_2 \dfrac{\partial u_2^{\ell+1}}{\partial \nu} = \phi & \text{on } \Gamma, \\ \sigma_1 \dfrac{\partial u_1^{\ell+1}}{\partial \nu} + \alpha u_1^{\ell+1} = \lambda_{1,N}^\ell & \text{on } \Sigma, \\ \sigma_2 \dfrac{\partial u_2^{\ell+1}}{\partial \nu} - \alpha u_2^{\ell+1} = \lambda_{2,N}^\ell & \text{on } \Sigma, \\ \lambda_{1,N}^{\ell+1} = \lambda_{2,N}^\ell + 2\alpha u_2^{\ell+1}, \\ \lambda_{2,N}^{\ell+1} = \lambda_{1,N}^\ell - 2\alpha u_1^{\ell+1}. \end{cases} \tag{2}$$

▶ General convergence proof is given by P-L. Lions in [4]

▶ $\alpha > 0$ is given parameter that can be optimized to improve the convergence rates [Japhet-Nataf-Rogier (1998), Gander (2006), Gander-Dubois (2015)].

▶ Similarly to above, we also apply the optimized Schwarz algorithm for solving the Dirichlet problem (D_σ).

4. A combined inversion method

Goal: At each gradient descent iteration the direct problems (N_σ) and (D_σ) are not solved exactly $\Rightarrow$ Take only one or a few steps of (2)

Difficulty

The domain decomposition changes at each gradient descent iteration k.
$\Rightarrow$ Ambiguity in defining

$$\lambda_{1,N}^{\ell+1}|_{\Sigma^{k+1}} = \lambda_{2,N}^\ell|_{\Sigma^k} + 2\alpha \, u_2^{\ell+1}|_{\Sigma^k}$$
$$\lambda_{2,N}^{\ell+1}|_{\Sigma^{k+1}} = \lambda_{1,N}^\ell|_{\Sigma^k} - 2\alpha \, u_1^{\ell+1}|_{\Sigma^k}$$

Algorithm 2: A combined inversion algorithm

- Fix the number of parameters $n \in \mathbb{N}^*$ that serve to define the interface.
- Consider an initial guess $R^0 \in (\mathbb{R}_+^*)^n$ with $\Sigma = \Sigma_{R^0} = \mathcal{T}_n(R^0)$.
- Initial boundary values: $\lambda_{i,N}(\Sigma) = 0, \; \lambda_{i,D}(\Sigma) = 0$ on $\Sigma, \; i = 1, 2$.
- k=0.

repeat *until $k \leq$ maximum number of iterations*

- Set $\lambda_{i,N}^0 = \lambda_{i,N}(\Sigma), \; \lambda_{i,D}^0 = \lambda_{i,D}(\Sigma)$ on $\Sigma, \; i = 1, 2$.
- Use L iterations of (2) to evaluate

$$u_{\sigma(R^k)} = N^L(\Sigma, \lambda_{1,N}^0, \lambda_{2,N}^0), \; \lambda_{i,N}^L(R^k) = \Lambda_{i,N}^L(\Sigma, \lambda_{1,N}^0, \lambda_{2,N}^0).$$

$$v_{\sigma(R^k)} = D^L(\Sigma, \lambda_{1,D}^0, \lambda_{2,D}^0), \; \lambda_{i,D}^L(R^k) = \Lambda_{i,D}^L(\Sigma, \lambda_{1,D}^0, \lambda_{2,D}^0).$$

- Evaluate $\dfrac{\partial \mathcal{J}}{\partial R_j}(R^k)$, for $j = 0, \dots, n-1$ where the boundary values are calculated using $u_{\sigma(R^k)_{|\Omega_1}}$ and $v_{\sigma(R^k)_{|\Omega_1}}$.
- Update $\Sigma = \mathcal{T}_n(R^{k+1})$ with $R_j^{k+1} := R_j^k - \tau \dfrac{\partial \mathcal{J}}{\partial R_j}(R^k)$, where $\tau > 0$ is chosen sufficiently small (A step adaptation can be incorporated).
- Update the interface values on S_j as

$$\lambda_{i,N}(\Sigma)(\hat{M}_j(t)) = \lambda_{i,N}^L(R^k)(t), \; t \in [0, 1],$$

$$\lambda_{i,D}(\Sigma)(\hat{M}_j(t)) = \lambda_{i,D}^L(R^k)(t), \; t \in [0, 1],$$

- $R^k = R^{k+1}$.
- $k = k + 1$.

end repeat

Local convergence result of Algorithm 2 for circular domains

Theorem [2]

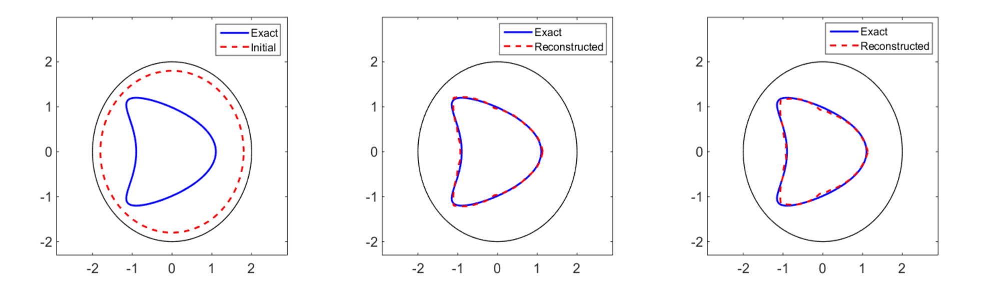
There exists $\delta_\alpha > 0$ such that Algorithm 2 with $L = 1$ is locally convergent for $\tau \in]0, \delta_\alpha[$.

Numerical experiments and validation

- Ω a open disk of center $(0, 0)$ and radius $r = 2$.
- $\sigma_1 = 1$, $\sigma_2 = 2$, $\alpha = 1$ and $\phi = \cos(\theta)$, $\theta \in [0, 2\pi]$.
- Let Σ be a kite defined by

$$\begin{cases} x(t) = \cos(t) + 0.5\cos(2t) - 0.4 \\ y(t) = 1.2\sin(t), \; t \in [0, 2\pi]. \end{cases} \tag{3}$$

- Using the parametrization (1) with $n = 19$. The initial guess is $R_j^0 = 1.8$.
- The gradient descent parameter $\tau = 0.05$ for both algorithms.



(a) Initial guess (b) Algorithm 1 (c) Algorithm 2

Figure 3: Comparison between Algorithm 1 and Algorithm 2 with $L = 1$ for the case of the kite parameterized by (3) and for noise free data. The exact shape and initial guess are shown in Figure (a).

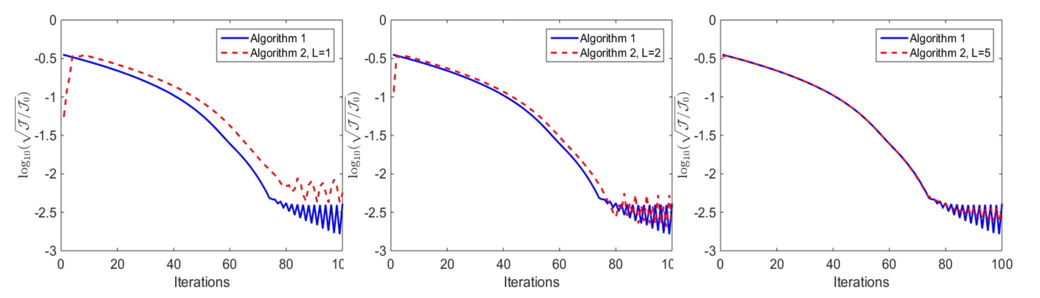


Figure 4: Comparison of the evolution of $\log_{10}(\sqrt{\mathcal{J}/\mathcal{J}_0})$ between Algorithm 1 and Algorithm 2 with $L = 1$ (left), $L = 2$ (middle) and $L = 5$ (right) for the example shown in Figure 3.

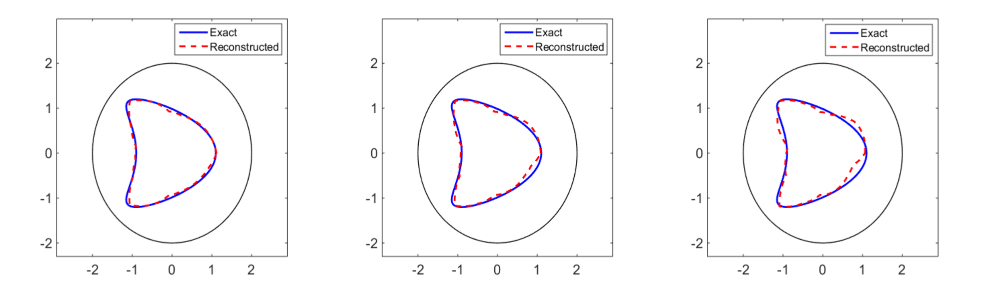


Figure 5: Reconstructions obtained by Algorithm 2 with $L = 1$ for the example discussed in Figure 3 but for noisy data with noise level $\epsilon = 1\%$ (left), $\epsilon = 3\%$ (middle) and $\epsilon = 5\%$ (right).

References

[1] L. Afraites, M. Dambrine, and D. Kateb. Shape methods for the transmission problem with a single measurement. *Numer. Funct. Anal. Optim.*, 519-551, 2007.

[2] S. Chaabane, H. Haddar, and R. Jerbi. A combination of kohn-vogelius and DDM methods for a geometrical inverse problem. *Submitted*, 2023.

[3] V. Dolean, P. Jolivet, and F. Nataf. An introduction to domain decomposition methods. *SIAM, Philadelphia*, 2015.

[4] P-L. Lions. On the schwarz alternating method iii. A variant for nonoverlapping subdomains, in: Third international symposium on domain decomposition methods for partial differential equations. *SIAM*, 202-223, 1990.