Identification of a local perturbation in unknown periodic layers

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Introduction

We consider the inverse scattering problem for the reconstruction of a **local perturbation in unknown periodic layers from near field measurements**.

Applications:

non-destructive testing of photonic structures, nanostructures, optical fibers...

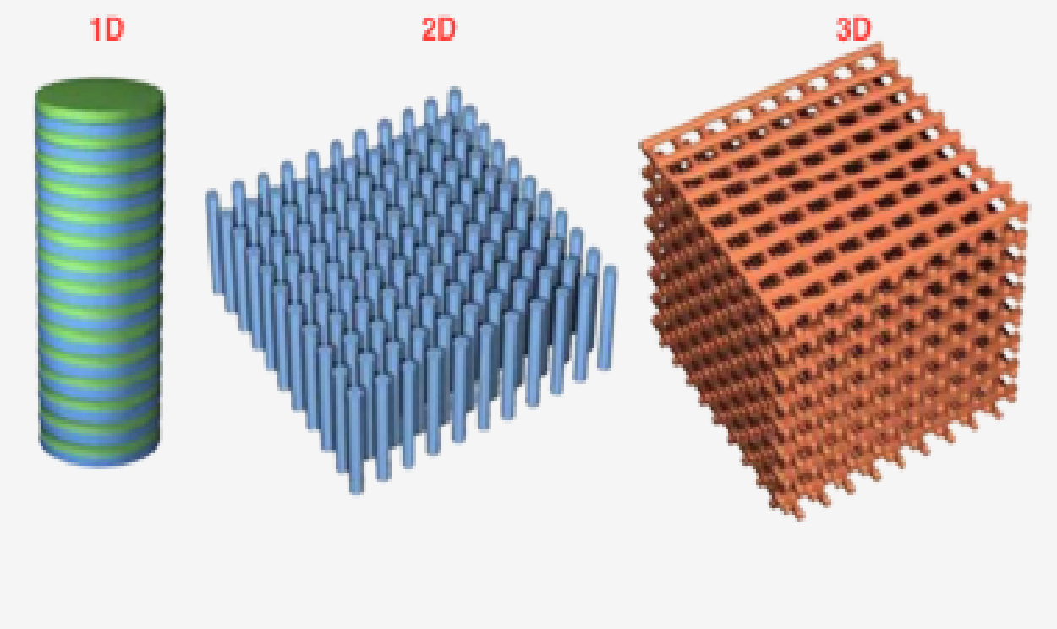


Figure: Illustration of a (a) 1D, (b) 2D, and (c) 3D photonic crystal structure

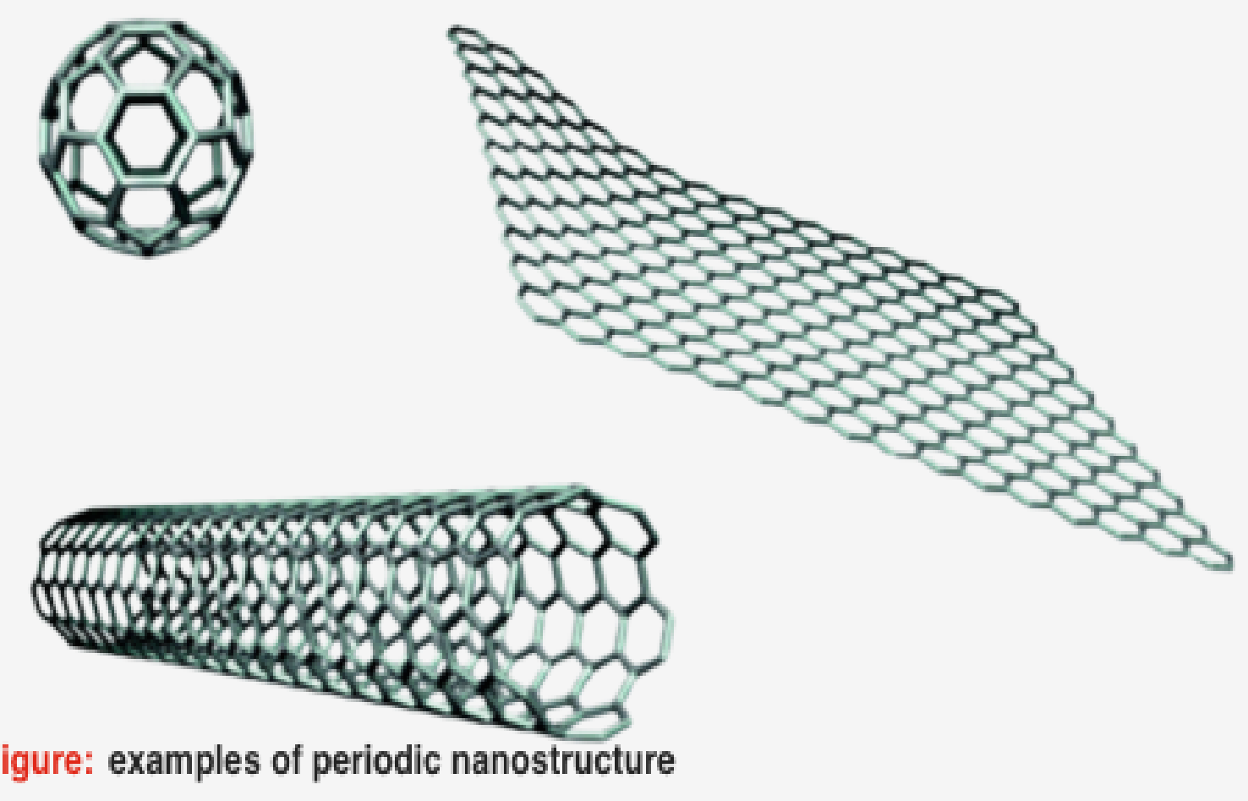
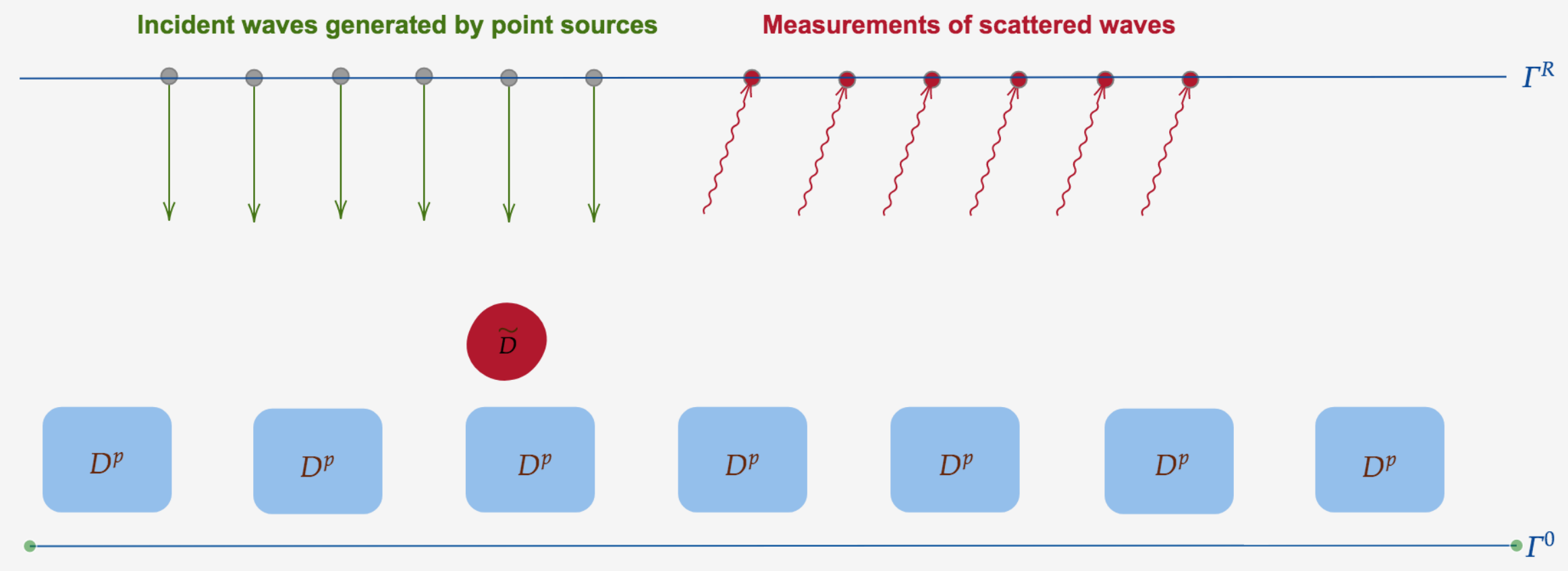


Figure: examples of periodic nanostructure

These photos are from Guldi, D. M., and Sgobba (2011), and S. Robinson and R. Nakkeeran (2013).

We consider an unbounded domain $\Omega := \mathbb{R} \times [0, \infty[$ delimited by $\Gamma^0 := \mathbb{R} \times \{0\}$,



Inverse problem From the measurements of scattered waves at $\Gamma^R := \mathbb{R} \times \{R\}$ for fixed frequency we identify:

- the periodic domain and the defect, only the periodic domain and only the perturbation. (H. Haddar and T.P. Nguyen (2017))

Difficulty: The perturbation brokes the periodicity

Direct Problem

Solution for an incident field $v \in L^2(D)$. We set $\Omega^R := \mathbb{R} \times [0, R]$.
Direct problem: Find $w \in H^2_{loc}(\Omega^R)$ satisfying

$$\boxed{(\mathcal{P}) \begin{cases} \Delta w + k^2 n w = k^2(1-n)v & \text{in } \Omega^R, \\ w = 0 & \text{on } \Gamma^0, \\ \text{+an upper going radiation condition} & \text{at } \Gamma^R, \end{cases}}$$

- $n_p(\cdot, x_2)$ is 2π-periodic and verifies $n_p = 1$ outside D^p.
- n is the perturbed refractive index verifies $n = n_p$ outside $\tilde{D}$.
- **Hypothesis:** Assume that $\{\Im m(n_p) > 0\}$ is not empty and that $\Im m(n) \geq 0$.
- $(\mathcal{P})$ is well posed (A. Konschin and A. Lechleiter (2019)).

Solution for a ξ-quasi periodic incident field v_ξ with period $2\pi M$, where $M \in \mathbb{N}$ s.t. $M \geq 2$.
Define

$$\tilde{H}^{s,\alpha}(\Omega^R) := \left\{ u \in H^s(\Omega^R) / \mathcal{J} u \in C^{0,\alpha}_\sharp(I, H^s_\xi(\Omega^R_0)) \right\}.$$

- $I := [0, 1]$, $\Omega^R_0 := [0, 2\pi M] \times [0, R]$ and $H^s_\xi(\Omega^R)$ the set of ξ-quasi periodic functions in $H^s_{loc}(\overline{\Omega^R})$ with period $2\pi M$.
- $\mathcal{J}$ is the Floquet-Bloch transfrom (isomorphism between $H^s(\Omega^R)$ and $L^2(I, H^s_\xi(\Omega^R_0))$.

We rewrite the solution of $(\mathcal{P})$ with $v = v_\xi$ as

$$w_\xi := w^p_\xi + \tilde{w}_\xi, \quad (*)$$

with

- $w^p_\xi \in H^1_\xi(\Omega^R_0)$ verifies the ξ-quasi periodic problem,
- $\tilde{w}_\xi \in \tilde{H}^1(\Omega^R)$ is a perturbed solution.

Inverse Problem: First result

- **Inverse problem for quasi-periodic Incident fields:** We consider $v_\xi = \overline{\Phi_{\xi,M}(y,\cdot)}$ where $\Phi_{\xi,M}(x,y) := (\mathcal{J}\Phi(\cdot,y))(\xi,x)$ for all $y \in \Gamma^R_0 := [0, 2\pi M] \times \{R\}$.

We measure: $(u^p_\xi + \mathcal{J}(\tilde{u}_\xi)(\xi,\cdot))|_{\Gamma^R_0}$ solution of $(*)$ with $v_\xi = \Phi_{\xi,M}(x,y)$ for all $y \in \Gamma^R_0$.

- **The ξ-quasi periodic near field operator:** $N_\xi : L^2_\xi(\Gamma^R) \to L^2_\xi(\Gamma^R)$

$$\boxed{N_\xi g_\xi(x) := \int_{\Gamma^R_0} g_\xi(y) u^{s,p}_\xi(x,y) ds(y) + \int_{\Gamma^R_0} g_\xi(y) \mathcal{J}(\tilde{u}^s_\xi(\cdot,y))(\xi,x) ds(y) =: N^p_\xi g_\xi + \tilde{N}^p_\xi g_\xi.}$$

For $g_\xi \in L^2_\xi(\Gamma^R)$, we define

$$I_\xi g_\xi := \left| (N^p_\xi g_\xi, g_\xi)_{L^2(\Gamma^R_0)} \right| + \left| (\tilde{N}^p_\xi g_\xi, g_\xi)_{L^2(\Gamma^R_0)} \right|.$$

- **The GLSM method:**

Let $c(\alpha) > 0$ verifying $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$. We introduce

$$J^\alpha_\xi(\phi; g_\xi) := \alpha I_\xi(g_\xi) + \|(N^p_\xi + \tilde{N}_\xi)g_\xi - \phi\|^2, \quad j^\alpha_\xi(\phi) := \inf_{g_\xi \in L^2_\xi(\Gamma^R)} J^\alpha_\xi(\phi; g_\xi).$$

Let us denote by $\Phi_\xi(x,y)$ the ξ-quasi-periodic with period 2π.

Theorem 1: Consider $z \in \Omega^R$, and let $g^{\alpha,M}_\xi, g^\alpha_\xi \in L^2_\xi(\Gamma^R)$ such that

$$\begin{aligned} J^\alpha_\xi(\Phi_{\xi,M}(\cdot,z), g^{\alpha,M}_\xi(z)) &\leq j^\alpha_\xi(\Phi_{\xi,M}(\cdot,z)) + c(\alpha), \\ J^\alpha_\xi(\Phi_\xi(\cdot,z), g^\alpha_\xi(z)) &\leq j^\alpha_\xi(\Phi_\xi(\cdot,z)) + c(\alpha), \end{aligned}$$

Then

$$\boxed{z \in D \cap \Omega^R_0 \iff \lim_{\alpha \to 0} I_\xi(g^{\alpha,M}_\xi(z)) < \infty.} \qquad \boxed{z \in D^p \cap \Omega^R_0 \iff \lim_{\alpha \to 0} I_\xi(g^\alpha_\xi(z)) < \infty.}$$

Inverse Problem: Second result

- **Inverse problem for non-periodic incident fields:** We consider $v = \overline{\Phi(\cdot,y)}$ for all $y \in \Gamma^R$, with $\Phi(\cdot,y)$ is the fundamental solution for the Dirichlet half space problem.

We measure: $u^s(\cdot,\cdot)$ solution of $(\mathcal{P})$ for all $x, y \in \Gamma^R$.

- **The near field operator:** $N : \tilde{L}^2(\Gamma^R) \to \tilde{L}^2(\Gamma^R)$

$$\boxed{Ng(x) = \int_{\Gamma^R} u^s(x,y) g(y) ds(y).}$$

Consider $g \in \tilde{L}^2(\Gamma^R)$, we define

$$I(g) := \sup_{\xi \in I} I_\xi(\mathcal{J} g(\xi,\cdot)).$$

- **The GLSM method:**

Let $c(\alpha) > 0$ verifying $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$. We introduce

$$J_\alpha(\phi; g) := \alpha I(g) + \|Ng - \phi\|^2, \quad j_\alpha(\phi) = \inf_{g \in \tilde{L}^2(\Gamma^R)} J_\alpha(\phi; g).$$

Theorem 2: Consider $z \in \Omega^R$, and let $g^\alpha \in \tilde{L}^2(\Gamma^R)$ such that

$$J_\alpha(\Phi(\cdot,z), g^\alpha(z)) \leq j_\alpha(\Phi(\cdot,z)) + c(\alpha),$$

then

$$\boxed{z \in D \iff \lim_{\alpha \to 0} I(g^\alpha(z)) < \infty.}$$

From Theorem 1 and Theorem 2 one can design an indicator function that allows to directly reconstruct $\widetilde{D}$

Numerical examples

Reconstruction of the full domain and only the periodic domain (as given by Theorem 2)

- Period: $L = 2\pi$
- Number of periods: $M = 4$
- waves number $k = 3.5\pi/3.14$
- 1% added multiplicative random noise.