Stability Estimates For Potentials Associated With Time Harmonic Magnetic Schrödinger Operator

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1- Context

- **Goal:** Prove logarithmic-type estimates for retrieving the magnetic fields and electric potentials from the near field or far field maps.
- **Approach:** Using techniques from similar results for inhomogeneous inverse scattering problems based on the use of geometrical optic solutions.

3- The inverse problems

We define **the near field operator** $\mathcal{N}_{A,q}: L^2(\partial B) \to L^2(\partial B)$, as

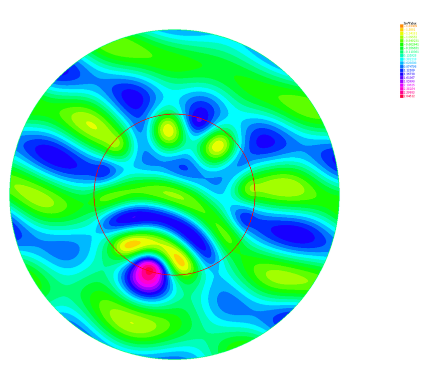
$$\mathcal{N}_{A,q}h(x) := \int_{\partial B} u^s_{A,q}(x,y)h(y)\,ds(y), \quad x \in \partial B.$$

Representing $u^s_{A,q}(\cdot,d)$, $d \in \mathbb{S}^2$ in terms of the outgoing fundamental solution of $\Delta + k^2$, it follows that as $|x| \to \infty$

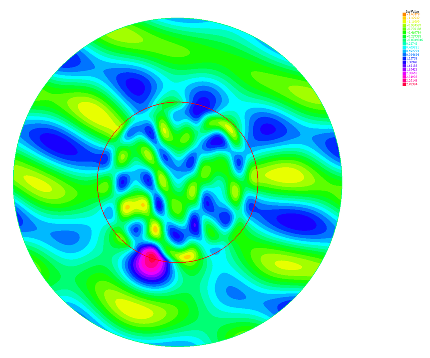
$$u^s_{A,q}(x,d) = \frac{e^{ik|x|}}{|x|}\left(u^\infty_{A,q}(\hat{x},d) + O\left(\frac{1}{|x|}\right)\right), \quad \hat{x} = \frac{x}{|x|},$$

where $u^\infty_{A,q}(\hat{x},d)$ is defined to be **the far field pattern**.

$\Rightarrow$ Due the Gauge invariance, the magnetic potential A cannot be uniquely determined from near or far field measurements outside B.



Total field $u_{A,q}(\cdot,d)$



Total field $u_{A+\nabla\varphi,q}(\cdot,d)$

where $\varphi \in W^{1,\infty}(\mathbb{R}^3)$ with support compactly embedded in B.

- Let

$$\operatorname{curl} A := \frac{1}{2}\sum_{i,j=1}^{3}\left(\partial_{x_i}a_j - \partial_{x_j}a_i\right)dx_j \wedge dx_i, \quad A = (a_j)_{1\le j\le 3}.$$

The inverse problem in the near field setting: is to determine $curl(A)$ and q from the knowledge of the near field operator $\mathcal{N}_{A,q}$.

The inverse problem in the far field setting: is to determine $curl(A)$ and q from the far field pattern $u^\infty_{A,q}(\hat{x},d)$, $\forall \hat{x}, d \in \mathbb{S}^2$.

4- The Classes of admissible A and q

Let $M > 0$, $\sigma > 0$ and $\gamma > 0$ be given.

$$\mathcal{A}_\sigma(M) := \left\{\operatorname{Supp}(A) \subset D,\ ||A||_{W^{2,\infty}(D)} \le M, \text{ and } \int_{\mathbb{R}^3}(1+|\xi|^2)^{\sigma/2}|\widehat{\operatorname{curl} A}(\xi)|d\xi \le M\right\},$$

$$\mathcal{Q}_\gamma(M) := \left\{\Im(q) \ge 0,\ \operatorname{Supp}(q) \subset D,\ ||q||_{L^\infty(D)} \le M \text{ and } \int_{\mathbb{R}^3}(1+|\xi|^2)^{\gamma/2}|\hat{q}(\xi)|d\xi \le M\right\}.$$

7- Stability estimates from far field pattern

Theorem 2

Let $M > 0$, $\sigma > 0$, $\gamma > 0$, and $\epsilon > 0$. Then there exist two constants $C > 0$ and $\delta > 0$ such that for all $(A_j, q_j) \in \mathcal{A}_\sigma(M) \times \mathcal{Q}_\gamma(M)$, $j = 1,2$, verifying $||u^\infty_{A_1,q_1} - u^\infty_{A_2,q_2}||_{L^2(\partial B\times\partial B)} < \delta$ we have

$$||\operatorname{curl}(A_1) - \operatorname{curl}(A_2)||_{L^\infty(D)} \le C|\log(\kappa)|^{-\frac{\sigma}{\sigma+3}+\epsilon},$$

and

$$||q_2 - q_1||_{L^\infty(D)} \le C|\log(\kappa)|^{-\frac{\gamma\sigma}{(\sigma+3)(2\gamma+3)}+\epsilon},$$

where $\kappa = ||u^\infty_{A_1,q_1} - u^\infty_{A_2,q_2}||_{L^2(\partial B\times\partial B)}$. Here C depends only on D, a, M, ϵ, σ, γ and δ.

Uniqueness: Let A_1 and $A_2 \in \mathcal{A}_\sigma(M)$, q_1 and $q_2 \in \mathcal{Q}_\gamma(M)$. If

$$u^\infty_{A_1,q_1}(\hat{x},d) = u^\infty_{A_2,q_2}(\hat{x},d), \quad \forall(\hat{x},d) \in \mathbb{S}^2 \times \mathbb{S}^2,$$

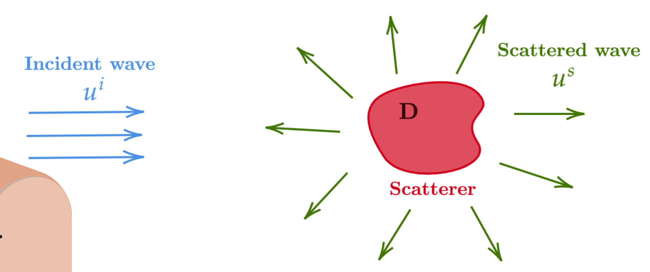
then $q_1 = q_2$ and $\operatorname{curl} A_1 = \operatorname{curl} A_2$ in D.

2- Introduction

We deal with a magnetic schrödinger operator in $3D$ case:

$$\mathcal{H}_{A,q}v = -\Delta v - i\operatorname{div}(Av) - iA\cdot\nabla v + (|A|^2 + q)v,$$

- Magnetic potential $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$, $\operatorname{supp}(A) \subset D$.
- Electric potential $q \in L^\infty(\mathbb{R}^3, \mathbb{C})$, $\operatorname{Im}(q) \ge 0$, $\operatorname{supp}(q) \subset D$.



- Let $B = B(0,a) \supset D$, $a > 0$ and $k > 0$ be the wave number.

Let $y \in \partial B$ be the location of a point source. **The direct scattering problem in the near field setting** is to find $u_{A,q}(\cdot,y)$ such that

$$\begin{cases} \mathcal{H}_{A,q}u_{A,q}(\cdot,y) - k^2u_{A,q}(\cdot,y) = \delta_y & \text{in } \mathbb{R}^3, \\ u_{A,q}(\cdot,y) = \Phi(\cdot,y) + u^s_{A,q}(\cdot,y) & \text{in } \mathbb{R}^3, \\ \lim_{r\to\infty} r\left(\partial_r u^s_{A,q} - iku^s_{A,q}\right) = 0 & r = |x|, \end{cases} \tag{1}$$

where $u^s_{A,q}(\cdot,y) \in H^2_{loc}(\mathbb{R}^3)$ and $\Phi(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$, $x \ne y$ is the fundamental solution of the Helmholtz equation.

Let $|y| \to \infty$ in the direction $-d$ with $d \in \mathbb{S}^2$. **The direct scattering problem in the far field setting** is to find $u_{A,q}(\cdot,d)$ such that

$$\begin{cases} \mathcal{H}_{A,q}u_{A,q}(\cdot,d) - k^2u_{A,q}(\cdot,d) = 0 & \text{in } \mathbb{R}^3, \\ u_{A,q}(\cdot,d) = u^i(\cdot,d) + u^s_{A,q}(\cdot,d) & \text{in } \mathbb{R}^3, \\ \lim_{r\to\infty} r\left(\partial_r u^s_{A,q} - iku^s_{A,q}\right) = 0 & r = |x|, \end{cases} \tag{2}$$

where $u^i(\cdot,d) = e^{ikx\cdot d}$ satisfies $\Delta u^i + k^2u^i = 0$ in $\mathbb{R}^3$ and $u^s_{A,q}(\cdot,d) \in H^2_{loc}(\mathbb{R}^3)$, $\forall d \in \mathbb{S}^2$.

5- Technical ingredients

- **Orthogonality identity** relates the difference of potentials to the difference of near-field operators.
- **Incorporation of geometric optics solutions**.
- **Relation between $\mathcal{N}_{A,q}$ and $u^\infty_{A,q}$:** Let $0 < \theta < 1$ be given. Then we have

$$||\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}|| \le \rho^2 \exp\left(-\left(-\ln\frac{||u^\infty_{A_1,q_1} - u^\infty_{A_2,q_2}||_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}}{\omega\rho}\right)^\theta\right),$$

where $\rho > 0$, $\omega > 0$ and $\mathcal{N}_{A_j,q_j}$, $j = 1,2$ denote the near field operators associated with $B = B(0,2a)$.

6- Stability estimates from near field operator

Theorem 1

Let $M > 0$, $\sigma > 0$ and $\gamma > 0$. Then there exists a constant $C > 0$ such that, for any $(A_j, q_j) \in \mathcal{A}_\sigma(M) \times \mathcal{Q}_\gamma(M)$, $j = 1,2$, we have

$$||\operatorname{curl}(A_1) - \operatorname{curl}(A_2)||_{L^\infty(D)} \le C(\kappa^{\frac{1}{2}} + |\log(\kappa)|^{-\frac{\sigma}{\sigma+3}}),$$

and

$$||q_2 - q_1||_{L^\infty(D)} \le C(\kappa^{\frac{1}{2}} + |\log(\kappa)|^{-\frac{\gamma\sigma}{(\sigma+3)(2\gamma+3)}}),$$

where $\kappa = ||\mathcal{N}_{A_1,q_1} - \mathcal{N}_{A_2,q_2}||$. Here C depends only on B, M, σ and γ.

Uniqueness: Let $A_1, A_2 \in \mathcal{A}_\sigma(M)$, $q_1, q_2 \in \mathcal{Q}_\gamma(M)$ and $B \supset D$. If

$$u^s_{A_1,q_1}(x,y) = u^s_{A_2,q_2}(x,y), \quad \forall(x,y) \in \partial B \times \partial B,$$

then $q_1 = q_2$ and $\operatorname{curl} A_1 = \operatorname{curl} A_2$ in D.

8- For more details

M. Bellassoued, H. Haddar and A. Labidi, *Stability estimate for an inverse problem for the time harmonic magnetic Schrödinger operator from the near and far field patterns.* SIMA, 2023.

9- Future Research

- The uniqueness of the reconstruction of the domain D for $q \ne 0$ and $A \ne 0$.
- Analysis of sampling methods for the reconstruction of the D support from the knowledge of the far-field data. Analysis of associated transmission interior problem.